DYNAMIC EQUILIBRIA IN HOMOGENOUS SYSTEMS AND A LABOR-MANAGED OLIGOPOLY MODEL

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ABSTRACT. In this paper we study homogenous differential equations, i.e. equations of the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ where $x \in U$, U is an open cone in a Banach space X, and for all $\mathbf{x} \in U$ and $\lambda > 0$ we have $\mathbf{F}(\lambda\mathbf{x}) = \lambda^r \mathbf{F}(\mathbf{x})$. Especially, we are interested in systems without a conventional equlibrium, which possess a dynamic equilibrium, i.e. an $\mathbf{x} \in X$ such that $\mathbf{F}(\mathbf{x}) = c\mathbf{x}$ where c is a scalar. We develop stability theory of dynamic equailibria, including an easy-to-verify spectral condition. As an application, we consider the dynamic model of a labor-managed oligopoly in the form of a system of differential equations. It has been known from our prior work that under natural, verifiable conditions (equivalent to asymptotically constant output of the industry) this system has an equilibrium which is globally stable. In the current paper we develop the notion of a dynamic equilibrium to handle the asymptotic behavior of the system when the total output of the industry shrinks or expands with time. We also consider two related general competition models derived from the labor-managed oligopoly model. One is an infinite-dimensional model which considers an infinite number of competing firms. Another is a stochastic model, in which the firms are drawn randomly from a large pool of firms. The stochastic model includes all other models considered as special cases.

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Key words and phrases. labor-managed,oligopoly,stability,dynamic,equilibria,homogenous.

This research has been supported in part by the National Science Foundation under grant no. DMS 9404419, and by the US Department of Energy, under contract W-7405-ENG-36.

1. Introduction

The main mathematical content of this paper is a systematic development of the theory of stability of homogenous systems. In particular, we focus on a new notion of an equilibrium, which we call a *dynamic equilibrium*, which generelizes the usual notion of an equilibrium. Dynamic equilibria occur naturally in this class of equations and are of the same practical significance from the point of view of applications as ordinary equilibria have for other classes of differential equations.

This article shall be of interest to mathematicians and scientists who may encounter homogenous systems in their research. The basic setup is explained in Section 1.1.

Our research of the subject was originally motivated by a particular model leading to a non-linear homogenous system of differential equation. In [4, 5, 7] the problem of local and global stability of a dynamic model of an economic system called labor-managed oligopoly was investigated. Such models have been studied in recent years [6]. A reader primarily interested in this model may consult these articles for details of the economic assumptions leading to the model and a discussion of Nash equilibria, a notion from game theory which plays an important role in the derivation of the model. In Section 1.3 we introduce the system of differential equations which is the model, and briefly summarize the economic assumptions.

The theory presented in our prior works relies upon an assumption on the parameters of the model which guarantees the existence of ordinary equilibria. This assumption leads to global asymptotic stability, by which the outputs of the individual firms go to values prescribed by the corresponding static Nash equilibrium.

Unfortunately, the same assumption implies that the economy cannot expand or shrink. From the point of view of the economic theory, it makes the model unrealistic. By introducing dynamic equilibria we were able to both increase the attractiveness of our model as well as obtain a satisfying mathematical theory which generalizes linear stability theory of ordinary equilibria.

In this paper the oligopoly model serves as both a motivation and an illustration of the techniques we developed for general non-linear homogenous equations. The main issue that our paper addresses is the fact that non-linear homogenous systems may exhibit stability even if they don't have conventional equilibria.

The notion of a dynamic equilibrium is naturally illustrated by the labor-managed oligopoly. If the economy consists of a number of competing firms making the same product, it is possible for the output of the economy and each individual firm to grow to infinity, while the *market share* of the individual firms goes to a definite limit.

1.1. **Homogenous Systems and Dynamic Equilibria.**

Let X be a (real) Banach space. A *cone* in X is a subset $C \subseteq X$ such that for for every $\lambda > 0$ we have $\lambda C = C$. An *open cone* is simply a cone which is an open set. A *convex cone* is a cone which is also a convex set. Equivalently, a set $C \subseteq X$ is a convex cone iff for any finite collection of numbers $\lambda_i > 0$ and points $\mathbf{x}_i \in C$, $i = 1, 2, \ldots, k$, also $\sum_{j=1}^{k} \lambda_i \mathbf{x}_i \in C$. A *ray* in X passing throuch $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ is the set

$$\mathscr{R}(\mathbf{x}) = \{\lambda \mathbf{x} \,:\, \lambda \in \mathbb{R}, \lambda > 0\}.$$

Let $U \subseteq X$ be an *open cone*. The main subject of our investigation is a system of differential equations that is homogenous[1], i.e. it is of the type

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1.1}$$

[1] This word is one of the most overloaded words in mathematics. Unfortunately, even in differential equations this word is used in a different sense: to denote a system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$, where $\mathbf{b} \equiv \mathbf{0}$.

where $\mathbf{F} : U \to X$ is of class C^1 and

$$\mathbf{F}(\lambda \mathbf{x}) = \lambda^r \mathbf{F}(\mathbf{x}) \tag{1.2}$$

for all $\lambda > 0$.[2] We will call equilibria which scale in time *dynamic equilibria*. The condition for $\mathbf{x}$ to be a dynamic equilibrium is:

$$\mathbf{F}(\mathbf{x}) = c \cdot \mathbf{x} \tag{1.3}$$

where c is a scalar constant (depending on $\mathbf{x}$), which we call the *expansion coefficient*. We note that if $\mathbf{x}$ is a dynamic equilibrium then for every $\lambda > 0$ the vector $\lambda\mathbf{x}$ is also a dynamic equilibrium. Hence, dynamic equilibria form a set which is a union of rays emanating from the origin. Moreover, one can verify that the flow line at a dynamic equilibrium $\mathbf{x}_0$ is actually the ray of dynamic equilibria and the solution with the initial condition $\mathbf{x}_0$ is given by

$$\mathbf{x}(t) \quad = \quad \begin{cases} \mathbf{x}_0\, e^{ct} & \text{if } r = 1\,, \\ \mathbf{x}_0\, [1 - c(r-1)t]^{-\frac{1}{r-1}} & \text{otherwise}\,, \end{cases} \tag{1.4}$$

assuming that $\mathbf{x}_0$ is a dynamic equilibrium and $\mathbf{F}(\mathbf{x}_0) = c\mathbf{x}_0$. Let us make several observations concerning the above formula. If $c = 0$, all the dynamic equilibria are the fixed points (oridinary equilibria). If $r > 1$ and $c > 0$, then the forward solution blows up to infinity along the ray of dynamic equilibria at finite time $t = \frac{1}{c(r-1)}$; if $r < 1$ and $c < 0$, then the forward solution shrinks to $\mathbf{0}$ along the ray of dynamic equilibria at finite time $t = \frac{1}{c(r-1)}$; in other cases, $c > 0$ or $c < 0$ the forward solution blows up to infinity or shrinks to $\mathbf{0}$ along the ray of dynamic equilibria as time tends to ∞.

Thus the flow line at a dynamic equilibrium either is a fixed point, or shrinks to $\mathbf{0}$, or blows up to ∞, depending on scalar c.

1.2. **Linear stability.** An ordinary equilibrium, when it is linearly stable, has the power to control the behavior of all nearby initial conditions. We recall that an equilibrium $\mathbf{x}$ is linearly stable for the equation (1.1) if the spectrum $\sigma(D\mathbf{F}(\mathbf{x}))$ is contained in the left halfplane of the complex plane, where $D\mathbf{F}(\mathbf{x})$ is the Frechét derivative of $\mathbf{F}$ at $\mathbf{x}$.

If $\mathbf{x}$ is "only" a dynamic equlibrium satisfying $\mathbf{F}(\mathbf{x}) = c \cdot \mathbf{x}$, it proves that there is a spectral condition that guarantees the control of a cone neighborhood of the ray $\mathscr{R}(\mathbf{x})$. Let us briefly explain the nature of this condition. First, the ray $\mathscr{R}(\mathbf{x})$ is invariant, and it contributes an eigenvalue $r \cdot c$ in $\sigma(D\mathbf{F}(\mathbf{x}))$. The condition is that:

(1) The eigenvalue $r \cdot c$ is simple (more precisely, a simple pole of the resolvent).
(2) There exists $\epsilon > 0$ such that the remainder of the spectrum must lie in the halfplane: [3]

$$\{z \in \mathbb{C} \,:\, \Re z \le \min(r \cdot c, c) - \epsilon\}.$$

This condition is a subject of Definition 3.1. The precise meaning of what we mean by "control" is given by Theorem 3.12, but roughly speaking it means that all solutions with initial conditions in a cone neighborhood of $\mathscr{R}(\mathbf{x})$ have the same asymptotics as given by formula (1.4). In particular, they do (do not) blow up in finite time iff the solution passing through the dynamic equilibrium does (does not) blow up.

1.3. **The labor-managed oligopoly model.** A labor-managed oligopoly consists of firms numbered from either 1 to n (the finite case) or from 1 to ∞ (the infinite case). Let us assume that these firms make a product whose quantity is measured by a single number, $x_i \in [0, \infty)$ (so called *single product industry*). In [4, 5, 7] both static and dynamic models of labor-managed oligopoly were derived and studied. For a derivation of the model from economic assumptions, the reader should consult one of these papers. The main assumptions of the model can be summarized by the following general principles:

[2] In the case of the labor-managed oligopoly model, $r = -2$.

[3] We note that this condition is stronger than merely the inclusion $\sigma(D\mathbf{F}(\mathbf{x}))\setminus\{r \cdot c\} \subseteq \{z \in \mathbb{C} \,:\, \Re z \le \min(r \cdot c, c) - \epsilon\}$. This is because $r \cdot c$ may not be a simple eigenvalue.

(1) The price function is $p(s) = b/s$; hence, it is inversly proportional to the total output of the industry

$$s = \sum_i x_i,$$

i.e. the price function is hyperbolic.

(2) The cost of production is a linear function of x_i, resulting from fixed costs and labor costs, assuming constant price of unit of labor.

(3) The objective of each firm is to maximize surplus per unit of labor.

(4) Every firm adjusts its output proportionally to its marginal profit.

The dynamic model of a labor-managed oligopoly is the following system of differential equations ($i = 1, 2, \ldots, n$ in the finite case and $i = 1, 2, \ldots$ in the infinite case):

$$\dot{x}_i = k_i \left(-\frac{b}{a_i\, s^2} + \frac{\beta_i}{a_i\, x_i^2} \right) \tag{1.5}$$

where $k_i > 0$ is a constant specified for each i, and $s = \sum_i x_i$. We will only consider $x_i > 0$. Moreover, in the infinite case we will add certain assumptions which imply that the sequence x_i is summable, in order for s to be finite.

In [7] the local existence and uniqueness of solutions for the infinite model was established for a suitable class of sequences x_i, for which there exists a constant $C < \infty$ such that for all i:

$$C^{-1} \leq \frac{\sqrt{b}}{\sqrt{\beta}_i} x_i \leq C.$$

Also, we have shown forward completeness of the system, i.e. for every initial condition the solution can be extended to the semi-infinite time interval $[0, \infty)$.

Finally, let us state the assumption used in all proofs:

$$\sum_i \sqrt{\beta_i} < \infty.$$

We note that β_i is the fixed cost of production, i.e. the cost of making $x_i = 0$ quantity of the product. In the finite case, this condition is trivially satisfied.

While we do not offer an interpretation for the quantity $\sum_i \sqrt{\beta_i}$, it is clear from purely mathematical considerations that this quantity plays an important role in this model. For instance, the necessary and sufficient condition for the system to possess an equilibrium is

$$\sum_i \sqrt{\beta_i} = \sqrt{b}.$$

The constant b determines the hyperbolic price function $p(s) = b/s$, i.e. the product costs $p(s)$ per its unit, assuming that the total output of the industry is s.

1.4. **A change of variables and a simplified system.** The first step in the proof is a change of coordinates. The new coordinates are given as follows:

$$y_i = \frac{\sqrt{b}}{\sqrt{\beta_i}} x_i. \tag{1.6}$$

In these new coordinates the model can be written as

$$\dot{y}_i = \frac{\sqrt{b}}{\sqrt{\beta_i}} \frac{k_i b}{a_i} \left(-\frac{1}{s^2} + \frac{1}{y_i^2} \right)$$

where $s = \sum_i \frac{\sqrt{\beta_i}}{\sqrt{b}} y_i$. Let us introduce the following notation:

$$\begin{aligned}\lambda_i &= \frac{\sqrt{b}}{\sqrt{\beta_i}} \frac{k_i b}{a_i},\\ \gamma_i &= \frac{\sqrt{\beta_i}}{\sqrt{b}}.\end{aligned}$$

It is an important assymption that

$$\sum_i \gamma_i < \infty$$

which is equivalent to $\sum_i \sqrt{\beta_i} < \infty$ in the original model (1.5).

The new system of differential equations can be written in a simplified form:

$$\left\{\begin{aligned}\dot{y}_i &= \lambda_i \left(-\frac{1}{s^2} + \frac{1}{y_i^2}\right),\\ s &= \sum_{i=1}^{n} \gamma_i y_i.\end{aligned}\right. \tag{1.7}$$

In most technical considerations we will use the simplified system (1.7). The exact translation of the results to the original coordinates can be accomplished with the help of (1.6).

2. Dynamic equilibria in the oligopoly model

In [4, 5, 7] it was shown that the necessary and sufficient condition for the system (1.5) to posses an equlibrium is

$$\sum_i \sqrt{\beta_i} = \sqrt{b}$$

Under this assumption, it can be shown that there is a unique ray of ordinary equilibria. Moreover, every solution of the system approaches one of these equilibria exponentially as time goes to infinity.

In the current section we will work towards weakening of the assumption that $\sum_i \sqrt{\beta_i} = \sqrt{b}$ by allowing the output vector to scale in time. Thus, we will allow the industry to expand or shrink. As the output of each individual firm expands or shrinks with the industry, the usual notion of an equilibrium is not applicable. However, if the output of the firms is measured as a fraction of the input of the industry, and this fraction approaches a definite limit over a long period of time, it makes sense to speak of a *dynamic equilibrium*.

Our results will be equally applicable to the cases of a finite and infinite number of firms. In order not to unnecessarily complicate notation, we will assume that i is an index varying either from 1 up to n in the case of n firms, or from 1 to infinity in the infinite case. Thus, notations like $\sum_i$ and $\sup_i$ will mean $\sum_{i=1}^{n}$ and $\sup_{1\leq i\leq n}$ in the finite case, and $\sum_{i=1}^{\infty}$ and $\sup_{1\leq i<\infty}$ in the finite case. When the distinction becomes important, we will comment on the difference between the two cases. One obvious difference is the necessity to deal with the Banach space ℓ^∞ in the infinite case, and the required functional-analytic nuances.

2.1. **The existence of dynamic equilibria for our oligopoly model.**

In our particular situation the condition of being a dynamic equilibrium reduces to:

$$k_i \left(-\frac{b}{a_i\, s^2} + \frac{\beta_i}{a_i\, x_i^2}\right) = c x_i, \qquad i = 1, 2, \ldots \text{(up to } n \text{ or } \infty) \tag{2.1}$$

Theorem 2.1. *(a) If $\sqrt{b} \leq \sum_i \sqrt{\beta_i}$ then there exists a unique ray of positive dynamic equilibria for which the output of the industry s does not shrink to* **0***, i.e. $c \geq 0$.*
(b) If $\sqrt{b} > \sum_i \sqrt{\beta_i}$ then there exists a constant $c_{crit} \in (-\infty, 0)$ such that

(1) *there are no dynamic equilibria with $c < c_{crit}$ or $c \geq 0$;*

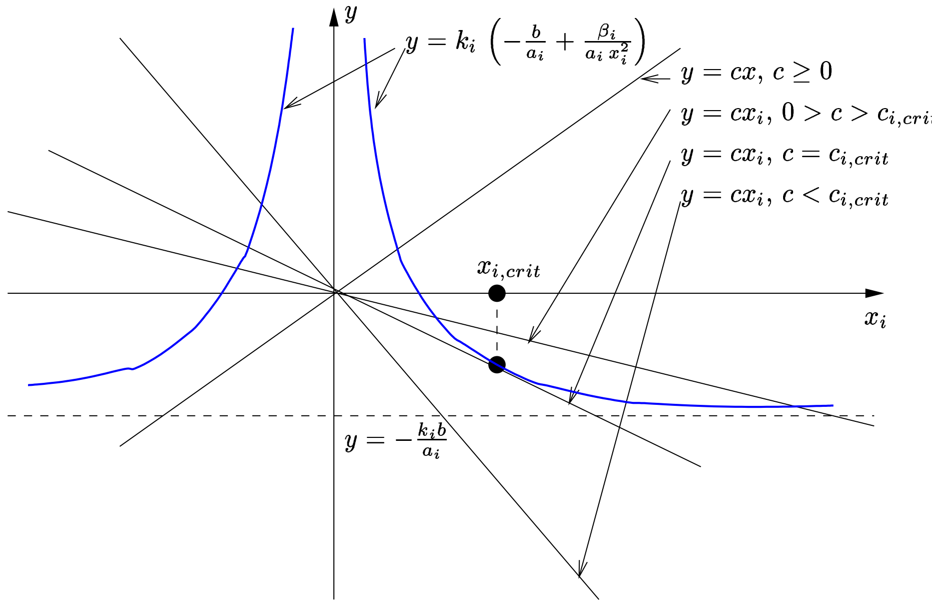

FIGURE 2.1. Determining dynamic equilibria.

(2) *for the number of firms* n *finite, there are at most* $3^n + 2^{n-1}$ *rays of positive dynamic equilibria with* $c \in [c_{crit}, 0)$*, for which the output of the industry* s *shrinks to* $\mathbf{0}$*, i.e.* $c < 0$*.*

Proof. (a) Let us assume that $\sqrt{b} \leq \sum_i \sqrt{\beta_i}$ and seek a dynamic equilibrium such that $s = 1$. Any other dynamic equilibrium can be scaled to satisfy this condition. The equation (2.1) for a fixed i and $c > 0$ has a unique solution $x_i > 0$. This is clear from studying the graphs of both sides of the equation. If $c = 0$, we obtain the solution $x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}}$ which we are already familiar with. However now $s = \sum_i x_i \geq 1$ for this solution. As c increases, all x_i strictly decrease, and go to 0. It is also true that x_i are continuous as functions of c, and so is s. By Intermediate Value Theorem, for some $c \geq 0$ we reach the point where $s = 1$ exactly. This solution is clearly unique because of the strict monotonicity of s. Thus the ray of positive dynamic equilibria with $c \geq 0$ is unique.
(b) Let us assume that $\sqrt{b} > \sum_i \sqrt{\beta_i}$ and seek a dynamic equilibrium such that $s = 1$. When sligthly decreasing c below 0, we note that the line $y = cx_i$ intersects the right branch of the graph of

$$y = k_i \left(-\frac{b}{a_i} + \frac{\beta_i}{a_i\, x_i^2}\right) \tag{2.2}$$

at two points (cf. Figure 2.1). This behavior persists until a critical value $x_i = c_{i,crit}$ is reached, when the line $y = cx_i$ becomes tangent to the graph of the functional relationship (2.2). The value of $c = c_{i,crit}$ and the corresponding value of $x_{i,crit}$ can be determined from the following system of equations:

$$cx_i = k_i\left(-\frac{b}{a_i}+\frac{\beta_i}{a_i\,x_i^2}\right) \tag{2.3}$$

$$c = -\frac{2k_i\beta_i}{a_i x_i^3} \tag{2.4}$$

We obtain:

$$c_{i,crit} = -\frac{2\,b^{\frac{3}{2}}\,k_i}{3\sqrt{3}\,a_i\,\sqrt{\beta_i}} \tag{2.5}$$

$$x_{i,crit} = \frac{\sqrt{3}\,\sqrt{\beta_i}}{\sqrt{b}} \tag{2.6}$$

(There is another solution with $x_i < 0$, differing just by the signs of x_i and c_i; it is discarded because we are interested in positive x_i only.)

When c drops to less than $c_{crit} = \sup_i c_{i,crit}$, there is no possibility of a dynamic equilibrium, as one of the equations (2.1) will cease to have a solution. On the other hand, when $c > c_{crit}$, we do have two values of $x_i(c)$ say, $x_i^-(c)$ and $x_i^+(c)$, with $x_i^-(c) < x_{i,crit} < x_i^+(c)$, which are solutions of the equation (2.3). Moreover, $x_i^-(c)$ is an decreasing function of c and $x_i^+(c)$ is an increasing function of c. Moreover, $\lim_{c\to 0-} x_i^+(c) = \infty$.

Clearly, each dynamic equilibrium with $c < 0$ is given by $x_i = x_i^{\pm}(c)$ for some choice of the sign for each i.

Let $\epsilon = (\epsilon_i)_{i=1}^{\infty}$ be any sequence of symbols $\pm$. For each such sequence, we define a function

$$s^\epsilon(c) \stackrel{\text{def}}{=} \sum_i x^\epsilon(c). \tag{2.7}$$

Only when all $\epsilon_i = -$ the function $s^\epsilon(c)$ is guaranteed to be defined and finite for $c = 0$, and decreasing as function of c. In all other cases $\lim_{c\to 0+} s^\epsilon(c) = \infty$. For each choice of ϵ there is an equilibrium of the form $x_i = x_i^{\epsilon_i}(c)$ iff

$$s^\epsilon(c) = 1. \tag{2.8}$$

Thus, the problem of finding dynamic equilibria reduces to finding solutions to 2^n equations (2.8). While we cannot in general predict the exact number of solutions of these equations, it is possible to give an upper bound. First, if all $\epsilon_i = -$ then the function $s^\epsilon(c)$ is a decreasing function of c and thus it can have only one solution. Moreover, the smalles value of it is attained at $c = 0$, when

$$x_i^-(c) = \frac{\sqrt{\beta_i}}{\sqrt{b}}.$$

Due to our assumption, $s^-(0) < 1$. The largest value we can count on is

$$\sum_i x_{i,crit} = \sqrt{3}\sum_i \frac{\sqrt{\beta_i}}{\sqrt{b}} = \sqrt{3}s^-(0)\,.$$

Hence, there are no solutions with all ϵ_i equal to a "$-$" if $\sum_i \frac{\sqrt{\beta_i}}{\sqrt{b}} < \frac{1}{\sqrt{3}}$.

To estimate the number of solutions for every other case, we experience a difficulty. The functions s^ϵ do not appear to have any special properties that would allow us to estimate the number of preimages of 1. For instance, functions $x_i^-(c)$ are convex, while $x_i^+(c)$ are not convex. Thus, the function $s^\epsilon(c)$ is generally non-convex. Convexity of a function implies that each value is assumed at most twice, and thus we would have an upper bound on the number of dynamic equilibria to be $2\cdot 2^n = 2^{n+1}$. Unfortunately, this idea fails.

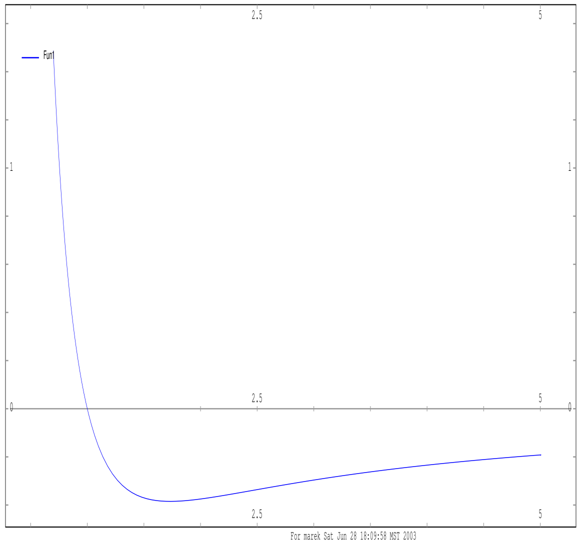

FIGURE 2.2. The graph of $f(x) = -\frac{1}{x} + \frac{1}{x^3}$ for some positive x.

We note that $x_i^{\pm}(c)$ are solutions of a cubic polynomial equation. Although explicit solutions can be written down, convexity does not follow easily from the resulting formulas. However, an implicit argument seems to be the only one available.

Let us state the relationship between c and $x_i^{\pm}(c)$ in the following form:

$$c = \frac{k_i}{a_i}\left(-\frac{b}{x_i} + \frac{\beta_i}{x_i^3}\right).$$

Let us define function $f_i(x_i) := -\frac{b}{x_i} + \frac{\beta_i}{x_i^3}$. The second derivative is $f_i''(x_i) = -\frac{2b}{x_i^3} + \frac{12\beta_i}{x_i^5}$. Hence, $f_i(x_i)$ has an inflection point at $x_i = \sqrt{6}\frac{\sqrt{\beta_i}}{\sqrt{b}}$. Using these functions, we can write

$$x_i^{\pm}(c) \in f_i^{-1}\left(\frac{a_i c}{k_i}\right).$$

In order to give an upper bound, we use a general algebraic argument, relying upon some facts from algebraic geometry. The function $s^\epsilon(c)$ is a sum of roots picked out amongst two roots of a cubic equation:

$$cx_i^3 + \frac{k_i b}{a_i}x_i^2 - \beta_i = 0. \tag{2.9}$$

If c is a scalar corresponding to a dynamic equilibrium, then there must be $c \geq c_{crit}$. For each of these c, there are at most 2^n possible solutions satisfying $s^\epsilon(c) = 1$. When $c = c_{crit}$, there is at least one i_0 such that the cubic equation (2.9) for x_{i_0} has only one positive roots and the rest of the cubic equations have at most 2 positive roots each. Thus the number of possible solutions to $s^\epsilon(c) = 1$ in this case is at most 2^{n-1}.

When $c > c_{crit}$, we have a longer argument. We will only be able to obtain an estimate on the total number of solutions c of the equations $s^\epsilon(c) = 1$, with variable ϵ, if we admit the third of the roots to the equation (2.9) in the range $x_i < 0$. Let us call this root $x_i^0(c)$. Thus, let us extend the range of symbols ϵ_i to $\{+, -, 0\}$, i.e. when we treat all roots of the cubic equation on equal footing. Then we can apply general algebraic considerations to find a bound on the number of solutions.

Let us consider the following function of c:

$$S(c) = \prod_{\epsilon}\left(s^\epsilon(c) - 1\right).$$

This function has two properties which allow us to apply it to finding our estimate.

First, every solution of every individual equation $s^\epsilon(c) = 1$ contributes a solution to the equation $S(c) = 0$. This is also true even counting multiplicities, i.e. if for a given $c_0 > c_{crit}$ there are k distinct sequences ϵ for which $s^\epsilon(c_0) = 1$ then $S(c)$ will have a multiple root at $c = c_0$ of multiplicity k, i.e. $|S(c)| = \mathrm{const}|c - c_0|^k + o(|c - c_0|^k)$ as $c \to c_0$.

The second important property of the function $S(c)$ is that it is a symmetric function in the roots of each of the cubic polynomials involved. Its total degree in all "variables" $x_i^{\epsilon_i}$ is 3^n. Therefore, by a famous theorem of Newton, it can be rewritten as a polynomial in symmetric functions of the roots of the cubic polynomials, i.e. the coefficients of the corresponding normalized polynomial. These coefficients are linear in $\frac{1}{c}$, and thus $S(c)$ is a polynomial in $\frac{1}{c}$ of degree 3^n, and thus it has a total of 3^n roots counting multiplicity and complex roots.

It is easy to see that the total number of pairs (ϵ, c) such that $s^\epsilon(c) = 1$ is not greater than $3^n + 2^{n-1}$. $\square$

Remark 2.2. We note that we do not have a general statement about the existence of at least one dynamic equilibrium in the case $\sqrt{b} > \sum_i \sqrt{\beta_i}$. However, our proof contains the methods of finding such equilibria, and clearly, they often exist. For instance, if we only slightly increase b above $\sum_i \sqrt{\beta_i}$, while keeping all the other parameters constant, then there will be a dynamic equilibrium of the type $x_i = x_i^-(c)$ in the notation of the above proof, with c negative.

Remark 2.3. One could ask whether there is a better bound for the number of dynamic equilibria in the case $\sqrt{b} > \sum_i \sqrt{\beta_i}$. For instance, is the number of roots $O(2^n)$ instead of $O(3^n)$?

Remark 2.4. In the case $\sqrt{b} > \sum_i \sqrt{\beta_i}$, when there are many coexisting attracting dynamic equilibria, it is natural to ask what the set of points are which are not attracted to any dynamic equilibria, i.e. that after passing to the factor system, do not converge to the corresponding ordinary equilibria. For instance, is this set fractal? The notion of a fractal basin boundary has been studied in many contexts in dynamical systems, and it would be interesting to see if this phenomenon occurrs in our model. Also, deterministic chaos in this region of parameters is not prohibited. Of course, just as we defined dynamic equilibria, we could consider *dynamic attractors* and *dynamic deterministic chaos*.

3. STABILITY OF EQUILIBRIA OF HOMOGENOUS SYSTEMS

In this section, we will discuss what conclusions one can draw for homogenous systems from the stability of equilibria of the reduced systems obtained by a change of variables.

3.1. **Semi-conjugacy for homogenous systems.** Let $\mathbf{F} : U \to X$ be a homogenous C^1 function defined on an open set U of a Banach space X (most interesting cases to us are $X = \mathbb{R}^n$ and $X = \ell^\infty$). Moreover, we assume that the function is *homogenous of degree* r, i.e. for all $\lambda > 0$ we have:

$$\mathbf{F}(\lambda \mathbf{x}) = \lambda^r \mathbf{F}(\mathbf{x}). \tag{3.1}$$

In particular, we assume that U is closed under the operation of multiplication by $\lambda > 0$, i.e. that it is a *cone*. Let us also assume that there is a bounded linear functional $\varphi : X \to \mathbb{R}$ which is positive on U. Then we can change variables in the system to essentially eliminate one variable.

Lemma 3.1. *Let $\Lambda_\lambda : U \to U$ be the linear scaling map with scale factor $\lambda > 0$, i.e.*

$$\Lambda_\lambda : U \ni \mathbf{x} \mapsto \lambda \mathbf{x} \in U.$$

The map $\mathbf{F} : U \to X$ is homogenous of degree r iff the map Λ_λ conjugates the system $\dot{\mathbf{x}} = \lambda^{-(r-1)}\mathbf{F}(\mathbf{x})$ to $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. In particular, $r = 1$ iff the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is invariant under scaling, i.e. any mapping Λ_λ with $\lambda > 0$ conjugates this system to itself.

Proof. Obvious. $\square$

Let us define

$$M_a \stackrel{\text{def}}{=} \{\mathbf{y} :\ \varphi(\mathbf{y}) = a\} = \varphi^{-1}(a)\,. \tag{3.2}$$

Theorem 3.2. *Let $\mathbf{F} : U \to X$ be C^1 and homogenous of any degree $r \in \mathbb{R}$, defined on an open cone $U \subseteq X$. Let $\pi : U \to U \cap M_1$ be a map given by:*

$$\pi(\mathbf{x}) = \frac{\mathbf{x}}{\varphi(\mathbf{x})}. \tag{3.3}$$

Let $\mathbf{G} : U \to X$ be defined by the formula:

$$\mathbf{G}(\mathbf{y}) \stackrel{\text{def}}{=} \varphi(\mathbf{y})\mathbf{F}(\mathbf{y}) - \varphi(\mathbf{F}(\mathbf{y}))\,\mathbf{y} \tag{3.4}$$

Then π is a semi-conjugacy of the system $\dot{\mathbf{x}} = \tilde{\mathbf{F}}(\mathbf{x})$, where $\tilde{\mathbf{F}} = \varphi^{-(r-1)}\mathbf{F}$, to the system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ on M_1. More generally, $\varphi \circ \mathbf{G} \equiv 0$. Thus $\mathbf{G}(\mathbf{y})$ is tangent to the affine subspace M_a which is invariant for every $a > 0$ with respect to the system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$. $\mathbf{G}$ can be considered as vector field on any of the affine subspaces $M_a \subset X$ of codimension 1.

Proof. It is easy to see that a line passing through the origin but not including the origin is mapped via (3.3) to a unique point in the new coordinate system.

Let us derive a differential equation for $\mathbf{y} = \pi(\mathbf{x})$. We have

$$\begin{aligned}
\frac{d\mathbf{y}}{dt} &= \frac{1}{\varphi(\mathbf{x})^2}\left[\varphi(\mathbf{x})\frac{d\mathbf{x}}{dt} - \varphi\left(\frac{d\mathbf{x}}{dt}\right)\mathbf{x}\right] \\
&= \frac{1}{\varphi(\mathbf{x})^2}\left[\varphi(\mathbf{x})\mathbf{F}(\mathbf{x}) - \varphi(\mathbf{F}(\mathbf{x}))\,\mathbf{x}\right] \\
&= \frac{1}{\varphi(\mathbf{x})^2}\left[\varphi(\varphi(\mathbf{x})\mathbf{y})\mathbf{F}(\varphi(\mathbf{x})\mathbf{y}) - \varphi(\mathbf{F}(\varphi(\mathbf{x})\mathbf{y}))\,\varphi(\mathbf{x})\mathbf{y}\right] \\
&= \varphi(\mathbf{x})^{r-1}\left[\varphi(\mathbf{y})\mathbf{F}(\mathbf{y}) - \varphi(\mathbf{F}(\mathbf{y}))\,\mathbf{y}\right]
\end{aligned}$$

For every $\mathbf{x} \in U$, this equation is equivalent to

$$D\pi(\mathbf{x})\mathbf{F}(\mathbf{x}) = \varphi(\mathbf{x})^{r-1}\mathbf{G}(\pi(\mathbf{x}))$$

Hence, the pair (π, φ^{r-1}) is an orbit semi-conjugacy of $\tilde{\mathbf{F}}$ and $\mathbf{G}$ as indicated in the statement of the theorem.

We easily verify that $\varphi(\mathbf{G}(\mathbf{y})) = 0$ for every $\mathbf{y} \in U$. Indeed,

$$\begin{aligned}
\varphi(\mathbf{G}(\mathbf{y})) &= \varphi\left(\varphi(\mathbf{y})\mathbf{F}(\mathbf{y}) - \varphi(\mathbf{F}(\mathbf{y}))\,\mathbf{y}\right) \\
&= \varphi(\mathbf{y})\varphi(\mathbf{F}(\mathbf{y})) - \varphi(\mathbf{F}(\mathbf{y}))\,\varphi(\mathbf{y}) = 0\,.
\end{aligned}$$

The tangent space of M_a is the nullspace of φ. Hence $\mathbf{G}(\mathbf{y})$ is tangent to M_a for any $\mathbf{y} \in M_a$. Moreover,

$$\frac{d\varphi(\mathbf{y})}{dt} = \varphi(\mathbf{G}(\mathbf{y})) = 0$$

Therefore, M_a is invariant with respect to $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$. It is obvious that M_a is of codimension 1. □

Remark 3.3. We may rephrase the main statement of Theorem 3.2 as follows: If in the system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ we introduce new time τ related to the old time t via the relation:

$$d\tau = \varphi(\mathbf{x})^{r-1}\,dt. \tag{3.5}$$

then the resulting system is $d\mathbf{x}/d\tau = \tilde{\mathbf{F}}(\mathbf{x})$. The map π is a semi-conjugacy of the latter to the system $d\mathbf{y}/d\tau = \mathbf{G}(\mathbf{y})$.

Remark 3.4. The map $\mathbf{G}$ is homogeneous, of degree $r + 1$, if considered on the domain of $\mathbf{F}$. The map $\tilde{\mathbf{F}}$ in the statement of Theorem 3.2 is homogenous of degree 1, i.e. it is scale invariant.

The degree of homogeneity of a system can be changed arbitrarily by changing time, i.e. multipltying $\mathbf{F}$ by a suitable power of φ.

Lemma 3.5. *Under the assumptions of Theorem 3.2, for every $a > 0$ the systems $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ restricted to M_1 is conjugate to the system $\dot{\mathbf{y}} = a^r\mathbf{G}(\mathbf{y})$ restricted to M_a, and the conjugacy is Λ_a.*

Proof. This is true in view of $\Lambda_a(M_1) = M_a$, the fact that $\mathbf{G}$ is homogenous of degree $r+1$ and Lemma 3.1. □

The following equation we will call the *reduced equation*:

$$\frac{d\mathbf{y}}{d\tau} = \varphi(\mathbf{y})\mathbf{F}(\mathbf{y}) - \varphi(\mathbf{F}(\mathbf{y}))\,\mathbf{y} = \mathbf{G}(\mathbf{y}). \tag{3.6}$$

We note that for $\mathbf{y} \in M_1$

$$\mathbf{G}(\mathbf{y}) = \mathbf{F}(\mathbf{y}) - \varphi(\mathbf{F}(\mathbf{y}))\mathbf{y}, \tag{3.7}$$

i.e. $\mathbf{G}(\mathbf{y})$ is the projection of $\mathbf{F}(\mathbf{y})$ onto the tangent space of M_1 (i.e. the nullspace of φ), when $\mathbf{F}(\mathbf{y})$ is decomposed along the direction of $\mathbf{y}$ and the tangent space of M_1.

Lemma 3.6. *A point $\bar{\mathbf{y}} \in U$ is a dynamic equilibrium of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ iff it is an equilibrium of the system on M_a given by the equation $\dot{\mathbf{y}} = \mathbf{G}_a(\mathbf{y})$, where $a = \varphi(\bar{\mathbf{y}})$.*

Proof. Assume that $\bar{\mathbf{y}} \in U$ is a dynamic equilibrium of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, i.e., $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$ for some c. Then

$$\mathbf{G}(\bar{\mathbf{y}}) = \varphi(\bar{\mathbf{y}})\mathbf{F}(\bar{\mathbf{y}}) - \varphi(\mathbf{F}(\bar{\mathbf{y}}))\bar{\mathbf{y}} = \varphi(\bar{\mathbf{y}})c\bar{\mathbf{y}} - \varphi(c\bar{\mathbf{y}})\bar{\mathbf{y}} = ac\bar{\mathbf{y}} - ca\bar{\mathbf{y}} = \mathbf{0}\,,$$

where $a = \varphi(\bar{\mathbf{y}})$. Since $\bar{\mathbf{y}}) \in M_a$, thus $\bar{\mathbf{y}}$ is an equilibrium of $\dot{\mathbf{y}} = \mathbf{G}_a(\mathbf{y})$ on M_a.

On the other hand, assume that $\bar{\mathbf{y}} \in U$ is an equilibrium of the system $\dot{\mathbf{y}} = \mathbf{G}_a(\mathbf{y})$ on M_a, i.e., $\mathbf{G}(\bar{\mathbf{y}}) = \mathbf{0}$. Then

$$\varphi(\bar{\mathbf{y}})\mathbf{F}(\bar{\mathbf{y}}) - \varphi(\mathbf{F}(\bar{\mathbf{y}}))\bar{\mathbf{y}} = \mathbf{0}\,,$$
$$\mathbf{F}(\bar{\mathbf{y}}) = \frac{\varphi(\mathbf{F}(\bar{\mathbf{y}}))}{\varphi(\bar{\mathbf{y}})}\bar{\mathbf{y}}\,.$$

Thus $\bar{\mathbf{y}}$ is a dynamic equilibrium of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with the scalar $\bar{c} = \frac{\varphi(\mathbf{F}(\bar{\mathbf{y}}))}{\varphi(\bar{\mathbf{y}})}$. □

3.2. **Scaling properties along rays of dynamic equilibria.** Let $\bar{\mathbf{y}}$ be an equilibrium of the equation $d\mathbf{y}/d\tau = \mathbf{G}(\mathbf{y})$ and $\varphi(\bar{\mathbf{y}}) = 1$ with a unique scalar $\bar{c}$, i.e. $\mathbf{F}(\bar{\mathbf{y}}) = \bar{c}\,\bar{\mathbf{y}}$. We note that $\bar{c}$ varies even within the same ray of dynamic equilibria. Indeed,

$$\mathbf{F}(\lambda\bar{\mathbf{y}}) = \lambda^r\mathbf{F}(\bar{\mathbf{y}}) = \lambda^r\bar{c}\bar{\mathbf{y}} = (\lambda^{r-1}\bar{c})(\lambda\bar{\mathbf{y}}).$$

If $\Gamma = \mathbb{R}^+\bar{\mathbf{y}} = \{\lambda\bar{\mathbf{y}} : \lambda \in \mathbb{R}^+\}$ is the ray containing $\bar{\mathbf{y}}$ then there exists a unique homogenous function $c : \Gamma \to \mathbb{R}$ such that for every $\mathbf{y} \in \Gamma$

$$\mathbf{F}(\mathbf{y}) = c(\mathbf{y})\mathbf{y}.$$

Specifically, $c(\lambda\bar{\mathbf{y}}) = \lambda^{r-1}\bar{c}$. Also, $c(\bar{\mathbf{y}}) = \bar{c}$.

Let us consider the relationship between the operators $D\mathbf{F}(\mathbf{y})$ computed for different points of the ray emanating from the origin. Differentiating both sides of the equation (3.1) with respect to $\mathbf{x}$, we obtain

$$D\mathbf{F}(\lambda\mathbf{y}) \;=\; \lambda^{r-1}D\mathbf{F}(\mathbf{y})\,. \tag{3.8}$$

Hence, the spectra of these operators are related as follows:

$$\sigma\left(D\mathbf{F}(\lambda\mathbf{y})\right) \;=\; \lambda^{r-1}\cdot\sigma\left(D\mathbf{F}(\mathbf{y})\right). \tag{3.9}$$

Also, differentiating both sides of the equation (3.1) with respect to λ at $\lambda = 1$, we obtain

$$D\mathbf{F}(\mathbf{y})\,\mathbf{y} = r\mathbf{F}(\mathbf{y})\,.$$

Then at $\mathbf{y} = \lambda\bar{\mathbf{y}}$, we have

$$D\mathbf{F}(\mathbf{y})\,\mathbf{y} = rc(\mathbf{y})\,\mathbf{y}\,.$$

Thus for every $\mathbf{y} \in \Gamma$, $rc(\mathbf{y})$ is the eigenvalue of $D\mathbf{F}(\mathbf{y})$ with eigenvector $\mathbf{y}$. Specifically, $r\bar{c}$ is the eigenvalue of $D\mathbf{F}(\bar{\mathbf{y}})$ with eigenvector $\bar{\mathbf{y}}$.

Denote $\sigma\left(D\mathbf{F}(\bar{\mathbf{y}})\right) = \bar{\sigma}_0 \cup \{r\bar{c}\}$ where $\bar{\sigma}_0$ and $\{r\bar{c}\}$ are disjoint. Then for every $\mathbf{y} \in \Gamma$ we have the following decomposition:

$$\sigma\left(D\mathbf{F}(\mathbf{y})\right) = \sigma_0(\mathbf{y}) \cup \{r\,c(\mathbf{y})\}.$$

where $\sigma_0 : \Gamma \to 2^{\mathbb{C}}$ is a homogenous set-valued function of degree $r-1$, i.e.

$$\sigma_0(\lambda\mathbf{y}) = \lambda^{r-1}\sigma_0(\mathbf{y}).$$

Clearly, σ_0 is given by the formula $\sigma_0(\lambda\bar{\mathbf{y}}) = \lambda^{r-1}\bar{\sigma}_0$. Since $c(\lambda\bar{\mathbf{y}}) = \lambda^{r-1}\bar{c}$, we know that $\sigma_0(\mathbf{y})$ and $\{r\,c(\mathbf{y})\}$ are also disjoint.

3.3. **Linear stability theory of dynamic equilibria.** The theory of linear stability of dynamic equilibria can be developed easily. By direct differentiation we obtain:

$$D\mathbf{G}(\mathbf{y})\mathbf{h} = \varphi(\mathbf{h})\mathbf{F}(\mathbf{y}) + \varphi(\mathbf{y})\,D\mathbf{F}(\mathbf{y})\,\mathbf{h} - \varphi(D\,\mathbf{F}(\mathbf{y})\,\mathbf{h})\,\mathbf{y} - \varphi(\mathbf{F}(\mathbf{y}))\,\mathbf{h} \tag{3.10}$$

The eigenvalue problem for the operator $D\mathbf{G}(\mathbf{y})$ can be simplified by introducing the following linear operator which involves only two of the four terms in the above formula:

$$\mathbf{L}(\mathbf{y})\mathbf{h} = \varphi(\mathbf{y})\,D\mathbf{F}(\mathbf{y})\,\mathbf{h} - \varphi(D\,\mathbf{F}(\mathbf{y})\,\mathbf{h})\,\mathbf{y}. \tag{3.11}$$

We note that

$$\varphi(\mathbf{L}(\mathbf{y})\mathbf{h}) \;=\; \varphi(\mathbf{y})\,\varphi(D\mathbf{F}(\mathbf{y})\,\mathbf{h}) - \varphi(D\,\mathbf{F}(\mathbf{y})\,\mathbf{h})\,\varphi(\mathbf{y}) = 0.$$

Thus, the range of $\mathbf{L}(\mathbf{y})$ lies in the nullspace N of the linear functional φ, or, in other words, φ is an eigenvector with zero eigenvalue of the conjugate operator $\mathbf{L}(\mathbf{y})^*$.

Let us assume that $\bar{\mathbf{y}} \in U$ is an equilibrium of equation (3.6), that is, $\mathbf{G}(\bar{\mathbf{y}}) = 0$. We verify that $\mathbf{L}(\bar{\mathbf{y}})\bar{\mathbf{y}} = 0$, but this calculation uses homogeneity. Indeed, differentiating equation (3.1) over λ at $\lambda = 1$ we obtain the known Euler's identity:

$$D\mathbf{F}(\bar{\mathbf{y}})\bar{\mathbf{y}} = r\mathbf{F}(\bar{\mathbf{y}}). \tag{3.12}$$

Thus

$$\mathbf{L}(\bar{\mathbf{y}})\bar{\mathbf{y}} = \varphi(\bar{\mathbf{y}})r\mathbf{F}(\bar{\mathbf{y}}) - \varphi(r\mathbf{F}(\bar{\mathbf{y}}))\bar{\mathbf{y}} = r\left(\varphi(\bar{\mathbf{y}})\mathbf{F}(\bar{\mathbf{y}}) - \varphi(\mathbf{F}(\bar{\mathbf{y}}))\bar{\mathbf{y}}\right) = r\,\mathbf{G}(\bar{\mathbf{y}}) = 0. \tag{3.13}$$

Similarly, $D\mathbf{G}(\bar{\mathbf{y}})\bar{\mathbf{y}} = 0$. Indeed, in general it is true that

$$D\mathbf{G}(\mathbf{y})\mathbf{y} = \mathbf{G}(\mathbf{y}) + \mathbf{L}(\mathbf{y})\mathbf{y}.$$

Hence, we can show our claim by plugging in $\bar{\mathbf{y}}$ for $\mathbf{y}$ and using the fact that $\mathbf{G}(\bar{\mathbf{y}}) = \mathbf{0}$, and (3.13). Thus, the line spanned by $\bar{\mathbf{y}}$ is an eigenspace of both operators. The vector $\bar{\mathbf{y}}$ is an eigenvector with eigenvalue 0.

In view of $\varphi \circ \mathbf{G} = 0$ and the chain rule we have

$$\varphi(D\mathbf{G}(\mathbf{y})\mathbf{h}) = D(\varphi\circ\mathbf{G})(\mathbf{y})\mathbf{h} = 0$$

for every vector $\mathbf{h}$. Also, from the definition of $\mathbf{L}$ it is clear that $\varphi(\mathbf{L}(\mathbf{y})\mathbf{h}) = 0$. Thus, the nullspace of φ is an invariant subspace of both $D\mathbf{G}(\mathbf{y})$ and $\mathbf{L}(\mathbf{y})$ for any $\mathbf{y}$.

Let N be the null-space of φ. Since $\varphi|_U \neq 0$ and equilibrium $\bar{\mathbf{y}} \in U$, we know that $\bar{\mathbf{y}} \notin N$. Let $\mathbf{y}$ be any point in X. In view of

$$\varphi\left(\mathbf{y} - \frac{\varphi(\mathbf{y})}{\varphi(\bar{\mathbf{y}})}\bar{\mathbf{y}}\right) = 0,$$

X is a direct sum of $\operatorname{span}\{\bar{\mathbf{y}}\}$ and the nullspace N of φ.

The block matrix of $\mathbf{L}(\bar{\mathbf{y}})$ with respect to the decomposition of the space into a direct sum of the span of $\bar{\mathbf{y}}$ and the nullspace N of φ is:

$$\mathbf{L}(\bar{\mathbf{y}}) = \left(\begin{array}{c|c} 0 & 0 \\ 0 & c\,\varphi(\bar{\mathbf{y}})\cdot Id_N + D\mathbf{G}(\bar{\mathbf{y}})|_N \end{array}\right) \tag{3.14}$$

In order to prove this claim, let us consider the difference of the operators $\mathbf{L}(\bar{\mathbf{y}})$ and $D\mathbf{G}(\bar{\mathbf{y}})$ which we will denote by $\mathbf{K}(\bar{\mathbf{y}})$. Thus, $\mathbf{K}(\bar{\mathbf{y}}) : X \to X$ is a linear operator. In view of the fact that for a dynamic equilibrium $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$, the difference is:

$$\begin{aligned}\mathbf{K}(\bar{\mathbf{y}})\mathbf{h} &= (\mathbf{L}(\bar{\mathbf{y}}) - D\mathbf{G}(\bar{\mathbf{y}}))\mathbf{h} \\ &= -\varphi(\mathbf{h})\,\mathbf{F}(\bar{\mathbf{y}}) + \varphi(\mathbf{F}(\bar{\mathbf{y}}))\,\mathbf{h} \\ &= -\varphi(\mathbf{h})\,c\,\bar{\mathbf{y}} + \varphi(c\,\bar{\mathbf{y}})\,\mathbf{h} \\ &= -c\,\varphi(\mathbf{h})\,\bar{\mathbf{y}} + c\,\varphi(\bar{\mathbf{y}})\,\mathbf{h}.\end{aligned}$$

If $\mathbf{h} = \bar{\mathbf{y}}$ then the right hand side is zero. This fact accounts for the right upper corner of the matrix (3.14) being zero. On the other hand, if $\mathbf{h} \in N$ then $\mathbf{K}(\bar{\mathbf{y}})\mathbf{h} = c\,\varphi(\bar{\mathbf{y}})\,\mathbf{h}$. Hence,

$$\mathbf{K}(\bar{\mathbf{y}})|_N = c\,\varphi(\bar{\mathbf{y}}) \cdot Id_N.$$

This formula implies the form of the right-lower entry of the matrix (3.14).

Finally, we will point out the relationship between the operator $D\mathbf{F}(\bar{\mathbf{y}})$, $\mathbf{L}(\bar{\mathbf{y}})$ and $D\mathbf{G}(\bar{\mathbf{y}})$. It will be convenient to make our statement in terms of the quotient space

$$Y = X/\operatorname{span}\{\bar{\mathbf{y}}\},$$

where X is the Banach space on which $\mathbf{F}$ is defined. As $\operatorname{span}\{\bar{\mathbf{y}}\}$ is an invariant subspace for all three operators, all three of them induce linear operators on Y. Let us adopt the notation $[A]^Y$ for the induced operator of a linear operator $A : X \to X$, for which $\bar{\mathbf{y}}$ is an eigenvector. From (3.14) it is clear that

$$c\,\varphi(\bar{\mathbf{y}}) \cdot Id_Y + [D\mathbf{G}(\bar{\mathbf{y}})]^Y = [\mathbf{L}(\bar{\mathbf{y}})]^Y.$$

In particular, these two operators have identical spectra. Also, directly from (3.10) it follows that for $\mathbf{y} = \bar{\mathbf{y}}$, using $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$ and assuming $\varphi(\bar{\mathbf{y}}) = 1$, as we have done in previous sections, we obtain:

$$\begin{aligned}D\mathbf{G}(\bar{\mathbf{y}})\mathbf{h} &= \varphi(\mathbf{h})\,c\,\bar{\mathbf{y}} + \varphi(\bar{\mathbf{y}})D\mathbf{F}(\bar{\mathbf{y}})\mathbf{h} - \varphi(D\mathbf{F}(\bar{\mathbf{y}})\mathbf{h})\bar{\mathbf{y}} - \varphi(c\,\bar{\mathbf{y}})\mathbf{h} \\ &= D\mathbf{F}(\bar{\mathbf{y}})\mathbf{h} - c\,\mathbf{h} + (\text{a scalar multiple of } \bar{\mathbf{y}})\,.\end{aligned}$$

Thus, in the case of $\varphi(\bar{\mathbf{y}}) = 1$,

$$c \cdot Id_Y + [D\mathbf{G}(\bar{\mathbf{y}})]^Y = [D\mathbf{F}(\bar{\mathbf{y}})]^Y = [\mathbf{L}(\bar{\mathbf{y}})]^Y.$$

In particular, the spectrum of $[D\mathbf{G}(\bar{\mathbf{y}})]^Y$ is the spectrum of $[D\mathbf{F}(\bar{\mathbf{y}})]^Y$ translated by $-c$. Thus, if the spectrum of $D\mathbf{F}(\bar{\mathbf{y}})$ is $\sigma_0 \cup \{rc\}$, and rc is a simple eigenvalue corresponding to the eigenvector $\bar{\mathbf{y}}$ then the spectrum of $[D\mathbf{G}(\bar{\mathbf{y}})]^Y$ is $\sigma_0 - c$. In view of the fact that the spectrum of $[D\mathbf{G}(\bar{\mathbf{y}})]^Y$ is the same as the spectrum of $D\mathbf{G}(\bar{\mathbf{y}})|_N$, we obtain:

Lemma 3.7. *The system $\dot{\mathbf{y}} = \mathbf{G}(\mathbf{y})$ is linearly stable at $\bar{\mathbf{y}}$ iff*

$$\sup_{z\in\sigma_0} \Re z < c. \tag{3.15}$$

Moreover, we have the following lemma which will prove to be useful later on:

Lemma 3.8. *Let us suppose that $(r-1)c < 0$. If*

$$\sup_{z\in\sigma_0} \Re z < rc \tag{3.16}$$

then there exists $\rho > 0$ such that $\rho + (r-1)c > 0$ (i.e. $\rho > |(r-1)c|$) and the spectrum of $D\mathbf{G}(\bar{\mathbf{y}})|_N$ (equal to $\sigma_0 - c$) is contained in the halfplane:

$$\{z \in \mathbb{C} : \Re z < -\rho\}.$$

The interest in the above lemma stems from the fact that in our economic model $r = -2$, and in the expanding industry case ($\sum_i \gamma_i > 1$) we have $c > 0$, and thus $(r-1)c < 0$. Furthermore, this case is the most delicate in the derivation of the asymptotic formulas of Theorem 3.12, our main result of this section.

Let us finish this section with the following definition:

Definition 3.1. *Let $\bar{\mathbf{y}}$ be a dynamic equilibrium of the homogenous system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

with expansion constant c, i.e. $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$. *This dynamic equilibrium is called* dynamically stable *iff it is a stable equilibrium of the reduced system (3.6).*

This dynamic equilibrium is called linearly dynamically stable *iff it it is linearly stable for the reduced system (3.6), or, equivalently, if there exists* $\epsilon > 0$ *such that the spectrum* $\sigma(D\mathbf{F}(\bar{\mathbf{y}}))$ *is contained in the halfplane* $\{z : \Re z \leq c - \epsilon\}$ *with the exception of a simple eigenvalue r c, corresponding to the eigenvector* $\bar{\mathbf{y}}$.

This dynamic equilibrium is called normally linearly dynamically stable *iff it it is linearly dynamically stable and either*

(1) $(r-1)c > 0$ *or*
(2) $(r-1)c < 0$ *and there exists* $\epsilon > 0$ *such that the spectrum* $\sigma(D\mathbf{G}(\bar{\mathbf{y}})) \subseteq \{z : \Re z \leq (r-1)c - \epsilon\}$ *or, equivalently, if such that the spectrum* $\sigma(D\mathbf{F}(\bar{\mathbf{y}}))$ *is contained in the halfplane* $\{z : \Re z \leq rc - \epsilon\}$ *with the exception of a simple eigenvalue r c, corresponding to the eigenvector* $\bar{\mathbf{y}}$.

Remark 3.9. This definition does not depend on the choice of the linear functional φ because every two choices produce systems equivalent up to a change of variables, and thus equivalent notions of stability.

3.4. **Explicit solution formulas.** In addition to the reduced equation (3.6) we can obtain the *complementary equation* which reflects the evolution of the scalar variable $\varphi(\mathbf{x})$:

$$\frac{d\varphi(\mathbf{x})}{dt} = \varphi\left(\frac{d\mathbf{x}}{dt}\right) = \varphi(\mathbf{F}(\mathbf{x})) = \varphi(\mathbf{F}(\varphi(\mathbf{x})\mathbf{y})) = \varphi(\mathbf{x})^r\varphi(\mathbf{F}(\mathbf{y})). \tag{3.17}$$

Thus, with respect to the new time τ, we have:

$$\frac{d\varphi(\mathbf{x})}{d\tau} = \varphi(\mathbf{x})\varphi(\mathbf{F}(\mathbf{y})). \tag{3.18}$$

The form of this equation invites introducing another variable:

$$\psi(\mathbf{x}) \stackrel{\text{def}}{=} \ln|\varphi(\mathbf{x})|$$

With respect to this variable, the complementary equation becomes

$$\frac{d\psi(\mathbf{x})}{d\tau} = \varphi(\mathbf{F}(\mathbf{y})), \tag{3.19}$$

i.e. the right-hand side is independent of $\mathbf{x}$.

Let us intoduce a scalar function:

$$g(\mathbf{y}) \stackrel{\text{def}}{=} \varphi(\mathbf{F}(\mathbf{y}))\,.$$

The behavior of the original system is completely captured by the following system of equations, considered on $U_1 \times \mathbb{R} \times \mathbb{R}$, where $U_1 = U \cap M_1$ is an open subset of M_1:

$$\left\{\begin{array}{rcl} \dfrac{d\mathbf{y}}{d\tau} & = & \mathbf{G}(\mathbf{y}), \\ \dfrac{dp}{d\tau} & = & g(\mathbf{y}) \\ \dfrac{dt}{d\tau} & = & \exp(-(r-1)p). \end{array}\right. \tag{3.20}$$

Clearly, if $\mathbf{x}(t)$ is the solution of the original equation $d\mathbf{x}/dt = F(\mathbf{x})$, and $t(\tau)$is computed by integration of the equation (3.5), and computing inverse function, i.e. first we find the function

$$\tau(t) = \tau_0 + \int_{t_0}^{t} \varphi(\mathbf{x}(t'))^{r-1}\,dt'$$

where τ_0, t_0 is arbitrarily chosen, then define $t(\tau)$ by computing the inverse function to $\tau(t)$. After this process is complete, the triple of functions

$$\begin{aligned} \mathbf{y}(\tau) &= \frac{\mathbf{x}(t(\tau))}{\varphi(\mathbf{x}(t(\tau)))}, \\ p(\tau) &= \ln|\varphi(\mathbf{x}(t(\tau)))|, \\ t &= t(\tau), \end{aligned}$$

is a solution of equations (3.20). And *vice versa*, given a solution $\mathbf{y}(\tau)$ of the first of the equations (3.20), we may obtain the value of $p(\tau)$ by integration:

$$p(\tau) = p(\tau_0) + \int_{\tau_0}^{\tau} g(\mathbf{y}(\tau'))\,d\tau' \tag{3.21}$$

for any $\tau_0 \in \mathbb{R}$. Moreover, the "old" time t can be also expressed in terms of the "new" time τ explicitly in view of $\frac{dt}{d\tau} = \exp(-(r-1)p)$. Thus, assuming that we have found $p(\tau)$,

$$t(\tau) = t(\tau_0) + \int_{\tau_0}^{\tau} \exp(-(r-1)p(\tau'))\,d\tau'. \tag{3.22}$$

The above consideration imply that the solution $\mathbf{x}(\tau)$ to the original equation, still expressed in terms of the "new" time τ, is:

$$\begin{aligned} \mathbf{x}(\tau) &= \exp(p(\tau))\mathbf{y}(\tau) \\ &= \exp\left(p(\tau_0) + \textstyle\int_{\tau_0}^{\tau} g(\mathbf{y}(\tau'))\,d\tau'\right)\mathbf{y}(\tau) \\ &= \varphi(\mathbf{x}(\tau_0))\exp\left(\textstyle\int_{\tau_0}^{\tau} g(\mathbf{y}(\tau'))\,d\tau'\right)\mathbf{y}(\tau). \end{aligned} \tag{3.23}$$

Thus, provided that the solution of the equation $d\mathbf{y}/d\tau = G(\mathbf{y})$ is known, all quantities of interest are expressed *in quadratures*, i.e. in terms of the operation of integration and algebraic operations. Thus, in both theoretical and practical terms, the system is "solved" when we found the dependence of $\mathbf{y}$ upon τ.

Let us assume that $\bar{\mathbf{y}} \in M_1$ (i.e. $\varphi(\bar{\mathbf{y}}) = 1$) is an equilibrium of the equation $d\mathbf{y}/d\tau = \mathbf{G}(\mathbf{y})$, i.e. that $\mathbf{G}(\bar{\mathbf{y}}) = \mathbf{0}$. Then from (3.4), we have

$$\mathbf{F}(\bar{\mathbf{y}}) = \frac{\varphi(\mathbf{F}(\bar{\mathbf{y}}))}{\varphi(\bar{\mathbf{y}})}\bar{\mathbf{y}}\,.$$

Thus, $\bar{\mathbf{y}}$ is a dynamic equilibrium of the equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$. *Vice versa*, if $\bar{\mathbf{y}} \in M_1$ and $\bar{\mathbf{y}}$ is a dynamic equilibrium of the equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ with $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$. Then $\varphi(\mathbf{F}(\bar{\mathbf{y}})) = c\varphi(\bar{\mathbf{y}})$ and thus $\mathbf{G}(\bar{\mathbf{y}}) = \mathbf{0}$. i.e., $\bar{\mathbf{y}}$ is an equilibrium of the equation $d\mathbf{y}/d\tau = \mathbf{G}(\mathbf{y})$.

We note that

$$g(\bar{\mathbf{y}}) = \varphi(\mathbf{F}(\bar{\mathbf{y}})) = \varphi(c\bar{\mathbf{y}}) = c\varphi(\bar{\mathbf{y}}) = c$$

as $\bar{\mathbf{y}} \in M_1$.

Let $\mathbf{x}(t)$ be the solution of the equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ with the initial condition $\mathbf{x}(0) = \bar{\mathbf{y}}$. It is possible to find this solution exactly using (3.23) because in this situation:

$$\mathbf{x}(\tau) = \varphi(\bar{\mathbf{y}})\exp\left(\int_0^{\tau} c\,d\tau'\right)\;\bar{\mathbf{y}} = \bar{\mathbf{y}}\;\exp(c\tau)\,.$$

Therefore, the solution along the equilibrium ray of $\bar{\mathbf{y}}$ is a simple exponential function.

From (3.21) we have

$$p(\tau) = p(0) + \int_0^{\tau} g(\mathbf{y}(\tau'))\,d\tau' = \ln\varphi(\bar{\mathbf{y}}) + \int_0^{\tau} g(\bar{\mathbf{y}})\,d\tau' = \ln 1 + \int_0^{\tau} c\,d\tau' = c\tau\,.$$

From (3.22), we have

$$
\begin{aligned}
t(\tau) &= \int_0^\tau \exp\left(-(r-1)c\tau'\right)\, d\tau' \\
&= \begin{cases} \dfrac{1}{(r-1)c}\left(1-\exp\left(-(r-1)c\tau\right)\right) & \text{if } r \neq 1, \\ \tau & \text{if } r = 1. \end{cases}
\end{aligned} \tag{3.24}
$$

Solving the inverse function to $t(\tau)$, we have

$$
\tau(t) = \begin{cases} -\dfrac{1}{(r-1)c}\ln[1-(r-1)ct] & \text{if } r \neq 1, \\ t & \text{if } r = 1. \end{cases} \tag{3.25}
$$

Then

$$
\mathbf{x}(t) = \begin{cases} [1-(r-1)ct]^{-\frac{1}{r-1}}\ \bar{\mathbf{y}} & \text{if } r \neq 1\,, \\ \exp(ct)\,\bar{\mathbf{y}} & \text{if } r = 1\,, \end{cases} \tag{3.26}
$$

which coincides with (1.4).

Let us comment on qualitative differences which happen in the solution depending on the values of c and r. The case of $r = 1$ is special, as the solution $\mathbf{x}(t)$ is defined on the entire line $(-\infty, \infty)$. In all other cases, the domain is a semi-infinite interval. We have:

$$
\text{The domain of } \mathbf{x}(t) = \begin{cases} \left(-\infty, \dfrac{1}{(r-1)c}\right) & \text{if } (r-1)c > 0, \\ \left(\dfrac{1}{(r-1)c}, \infty\right) & \text{if } (r-1)c < 0, \\ (-\infty, \infty) & \text{if } (r-1)c = 0. \end{cases}
$$

Of course, in each case 0 is in the domain of $\mathbf{x}(t)$.

When t approaches the special value $\frac{1}{(r-1)c}$, while remaining in the domain of $\mathbf{x}(t)$, the solution $\mathbf{x}(t)$ either goes to 0 (we will refer to this situation as *finite-time collapse*) or ∞ (this situation is often referred to as *finite-time blow-up*). From equation (3.26) we can see easily that the finite-time collapse happens when $r < 1$ and finite-time blowup corresponds to $r > 1$.

Similarly, at the infinite end of the domain of $\mathbf{x}(t)$ the system collapses when $r > 1$ and blows-up when $r < 1$.

In the remainder of this section we will determine the domain of $\mathbf{x}(t)$ for other initial conditions, which are not on a ray of dynamic equilibria. We will also study the asymptotic behavior of the solution at the ends of the domain. The major restriction on the class of these initial conditions for which we will be able to make these conclusions will be that they are attracted by some ray of dynamic equilibria.

3.5. **Asymptotic estimates of solutions.** Let us recall the definition of the *big-O* notation in sufficient generality to express the asymptotic properties of the solutions of our system. Let u be a Banach space valued function, and let v be a real-valued function, both defined on a set $D \subseteq T$, where T is a topological space. Let $t_0 \in T$. We say that $u(t) = O(v(t))$ as $t \to t_0$, if there exists an open set $U \subseteq T$ with $t_0 \in \overline{U}$ and a real constant $A \in [0, \infty)$ such that for all $t \in D \cap U$

$$
\|u(t)\| \leq A\, v(t).
$$

We note that if t_0 is not an accumulation point of D then this condition is automatically satisfied, and thus the notion of big-O is only useful when t_0 belongs to the closure of D.

On occasions, we will also use the *little-o* notation. With exactly the same setup as above, we say that $u(t) = o(v(t))$ a $t \to t_0$ if for every real constant $A \in (0, \infty)$ there exists an open set $U \subseteq T$ with $t_0 \in \overline{U}$

such that for all $t \in D \cap U$

$$\|u(t)\| \leq A\, v(t).$$

Thus, formally, the definition of little-o is obtained from the definition of big-O by changing the order of quantifiers.

Let us suppose that $\bar{\mathbf{y}}$ is an asymptotically stable equilibrium of the equation

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{G}(\mathbf{y})$$

and let $\mathbf{x}(t)$ be the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

with the initial condition $\mathbf{y}$, where $\mathbf{y}$ is sufficiently close to $\bar{\mathbf{y}}$. The purpose of this section is to derive the asymptotics for $\mathbf{x}(t)$ as $t \to \infty$ or as t tends to the blowup time.

There is a general mechanism by which such estimates can be obtained. Let us suppose that we have an asymptotics for the corresponding solution $\mathbf{y}(t)$, as $t \to \infty$. For instance, if $\bar{\mathbf{y}}$ is a linearly asymptotically stable equilibrium, there is an estimate:

$$\|\mathbf{y}(\tau) - \bar{\mathbf{y}}\| = O(\exp(-\epsilon\tau)),$$

where $\epsilon > 0$. In general, very precise estimates can be obtained for the difference $\mathbf{y}(\tau) - \bar{\mathbf{y}}$ as $\tau \to \infty$, based on the normal form theory. Such estimates are standard in the finite-dimensional case. In general, very long asymptotic expansions require high class of smoothness from $\mathbf{G}$.

The estimates are based on the explicit formulas: (3.22), (3.21) and (3.23). The main difficulty that we have to overcome is the use of different time τ and t.

Let us start with the following simple but useful lemma from asymptotic analysis:

Lemma 3.10. *Let $f : [0,\infty) \times T \to X$ be a Banach space valued function defined on the Cartesian product of $[0,\infty) \subseteq \overline{\mathbb{R}}$ and of a topological space T. Let $f(\tau, t) = O(g(t)\exp(-k\tau))$ as $(\tau, t) \to (\infty, t_0)$, where k is a constant and $g : T \to \mathbb{R}$ is a real-valued function and $t_0 \in T$ is arbitrary. Let us assume that for every $t \in T$ the function $[0,\infty) \ni \tau \mapsto f(\tau, t) \in X$ is absolutely integrable on any finite interval and let $q : [0,\infty) \times T \to X$ via*

$$q(\tau, t) = \int_0^{\tau} f(\tau', t)\, d\tau.$$

Then

(1) *If $k > 0$, there exists a function $C : T \to \mathbb{R}$ such that $C(t) = O(g(t))$ as $t \to t_0$ and, as $(\tau, t) \to (\infty, t_0)$,*

$$q(\tau, t) = C(t) + O(g(t)\exp(-k\tau)).$$

Moreover, if C_1 and C_2 are two functions, such that $q(\tau, t) = C_j(t) + O(g(t)\exp(-k\tau))$ for $j = 1, 2$ then there is an open neighborhood U of t_0 in T such that $C_1|_U = C_2|_U$.

(2) *If $k = 0$ then, as $(\tau, t) \to (\infty, t_0)$,*

$$q(\tau, t) = O(g(t)\tau).$$

(3) *If $k < 0$ then, as $(\tau, t) \to (\infty, t_0)$,*

$$q(\tau, t) = O(g(t)\exp(-k\tau)).$$

Proof. (1) Let $C(t) = \int_0^{\infty} f(\tau', t)\, d\tau'$. Obviously, the integral converges absolutely. Also, we have:

$$q(\tau, t) = C(t) - \int_{\tau}^{\infty} f(\tau', t)\, d\tau'.$$

In addition, if R is a constant such that $\|f(\tau,t)\| \leq Rg(t)\exp(-k\tau)$ on $[0,\infty)\times T$ then

$$\begin{aligned}\left|\int_\tau^\infty f(\tau',t)\,d\tau'\right| &\leq \int_\tau^\infty Rg(t)\cdot\exp(-k\tau')\,d\tau' \\ &= \frac{Rg(t)}{k}\exp(-k\tau) = O(g(t)\exp(-k\tau)).\end{aligned}$$

Thus, $q(\tau,t) = C(t) + O(g(t)\exp(-k\tau))$.

Let us suppose there are two functions $C_1(t)$ and $C_2(t)$ such that

$$q(\tau,t) = C_1(t) + O(g(t)\exp(-k\tau)) = C_2(t) + O(g(t)\exp(-k\tau)).$$

Then $C_1(t) - C_2(t) = O(g(t)\exp(-k\tau))$, i.e. there is a constant A and an open neighborhood U of t_0 in T, and a number $\tau_0 \in \mathbb{R}^+$ such that for all pairs $(\tau,t) \in (\tau_0,\infty)\times U$ we have

$$|C_1(t) - C_2(t)| \leq Ag(t)\exp(-k\tau)).$$

Letting $\tau \to \infty$, we obtain $C_1(t) = C_2(t)$ for all $t \in U$, and the proof of (1) is complete.

The proof of (2) and (3) is trivial. ☐

Let us formulate without a proof a well-known result from the theory of stability:

Lemma 3.11. *Let $\bar{\mathbf{y}}$ be a linearly asymptotically stable equilibrium of the equation $d\mathbf{y}/d\tau = \mathbf{G}(\mathbf{y})$, i.e $\mathbf{G}(\bar{\mathbf{y}}) = \mathbf{0}$ and the spectrum σ of the operator $D\mathbf{G}(\bar{\mathbf{y}})$ is in the left halfplane. Let $\rho > 0$ be such that $-\rho > \sup_{z\in\sigma}\Re z$. Then*

$$\mathbf{y}(t,\mathbf{y}_0) - \bar{\mathbf{y}} = O\left(\|\mathbf{y}_0 - \bar{\mathbf{y}}\|\exp(-\rho t)\right)$$

as $(t,\mathbf{y}_0) \to (\infty,\bar{\mathbf{y}})$.

Theorem 3.12. *Let Γ be a ray of dynamic equilibria of the equation $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ and let $\varphi : X \to \mathbb{R}$ be a linear functional strictly positive on Γ. Let $c : \Gamma \to \mathbb{R}$ be defined by the equation $\mathbf{F}(\mathbf{y}) = c(\mathbf{y})\mathbf{y}$. Let $\kappa : \Gamma \to \mathbb{R}$ be defined by:*

$$\kappa \stackrel{\text{def}}{=} (r-1)c.$$

Let us assume that for at least one $\mathbf{y} \in \Gamma$ the spectrum $\sigma(D\mathbf{F}(\mathbf{y})) = \sigma_0(\mathbf{y}) \cup \{r\,c(\mathbf{y})\}$ and $r\,c(\mathbf{y})$ is a simple eigenvalue with eigenvector $\mathbf{y}$, and there exists a positive constant $\rho(\mathbf{y})$ such that

$$\sup_{z\in\sigma_0(\mathbf{y})}\Re z < r\,c(\mathbf{y}) - \rho(\mathbf{y}). \tag{3.27}$$

In addition, if $\kappa(\mathbf{y}) < 0$, we will assume that

$$\kappa(\mathbf{y}) + \rho(\mathbf{y}) > 0. \tag{3.28}$$

By Section 3.2, if $\rho : \Gamma \to \mathbb{R}$ is extended to a homogenous function on Γ of degree $r-1$ then equations (3.27) and (3.28) are satisfied for all $\mathbf{y} \in \Gamma$. Let $\mathbf{\Phi} : U \to \Gamma$ be a linear projection given by $\mathbf{\Phi}(\mathbf{x}_0) = \varphi(\mathbf{x}_0)\bar{\mathbf{y}}$ where $\bar{\mathbf{y}} \in \Gamma$ is chosen so that $\varphi(\bar{\mathbf{y}}) = 1$. Let us extend c, κ and ρ to U by composing them with $\mathbf{\Phi}$, using the fact that $\mathbf{\Phi}|_\Gamma = Id_\Gamma$. Let

$$d(\mathbf{x}_0) = \|\mathbf{x}_0 - \mathbf{\Phi}(\mathbf{x}_0)\|$$

be the distance of $\mathbf{x}_0$ from the ray of dynamic equilibria measured along the level sets of φ.

There exists an open, convex cone W vertexed at $\mathbf{0}$ and containing Γ such that the solution to $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ with initial condition $\mathbf{x}_0 \in W$ satisfies the following asymptotic estimates:

1. *When $\kappa = 0$, there exist functions $t_0, c', \rho' : W \to \mathbb{R}$ of the initial condition $\mathbf{x}_0$ only, such that if $t \to \infty$ and $d(\mathbf{x}_0) \to 0$ then*

$$\mathbf{x}(t,\mathbf{x}_0) = \exp(c'(t-t_0))\mathbf{\Phi}(\mathbf{x}_0) + O\left(d(\mathbf{x}_0)\,\exp((c'-\rho')(t-t_0))\right). \tag{3.29}$$

(In the above formula we omitted the argument $\mathbf{x}_0$ in functions t_0, c' and ρ', in order not to excessively complicate our notation; we will follow this practice below.) The functions t_0 and c' are unique. The quantities t_0, $c'-c$, and $\rho'-\rho$, are all $O(d(\mathbf{x}_0))$ as $d(\mathbf{x}_0) \to 0$, in particular, t_0, c'

and ρ' are continuous extensions of functions $0, c, \rho : \Gamma \to \mathbb{R}$ respectively. If $c = 0$ then $c' = 0$ for all $\mathbf{x}_0$. If $c \neq 0$, we may choose

$$\rho' = \rho\frac{c'}{c}.$$

Functions c' and ρ' are homogenous of degree $r-1$ and t_0 is homogenous of degree $-(r-1)$.

(2) *If $\kappa \neq 0$, and either*
  (a) *$\kappa < 0$ and $t \to \infty$, or*
  (b) *$\kappa > 0$ and $t \nearrow t_0 + \frac{1}{\kappa}$,*
and $d(\mathbf{x}_0) \to 0$ then

$$\mathbf{x}(t, \mathbf{x}_0) \;=\; (1-\kappa(t-t_0))^{-\frac{1}{r-1}}\,\mathbf{\Phi}(\mathbf{x}_0) + O\left(d(\mathbf{x}_0)\,(1-\kappa(t-t_0))^{-\eta}\right) \tag{3.30}$$

where $\eta : W \to \mathbb{R}$ is given by

$$\eta = \frac{1}{r-1} - \frac{\rho}{\kappa} = \frac{c-\rho}{\kappa}.$$

The function $t_0 : W \to \mathbb{R}$ is unique, $t_0 = O(d(\mathbf{x}_0))$ as $d(\mathbf{x}_0) \to 0$ and t_0 is homogenous of degree $-(r-1)$.

In all cases, the first term of the asymptotic formula is dominant.

Proof. First we reduce the proof to the special case when $\varphi(\mathbf{x}_0) = 1$ and $\mathbf{x}_0 \to \bar{\mathbf{y}}$, where $\bar{\mathbf{y}} \in \Gamma$ such that $\varphi(\bar{\mathbf{y}}) = 1$. The reason why this is possible is that if $\mathbf{x}(t, \mathbf{x}_0)$ is a solution of our system then also $t \mapsto \lambda\mathbf{x}(\lambda^{r-1}t, \lambda\mathbf{x}_0)$ is a solution. Due to extreme care in formulating our theorem, all conclusions that hold for one of these solutions, also hold for the other. More precisely, we will define functions t_0, c' and ρ' on a convex neighborhood of $\bar{\mathbf{y}}$ in M_1 and then extend them to a convex cone W by homogeneity.

During the course of the proof we will consider a number of asymptotic estimates involving the initial condition $\mathbf{x}_0$, of the type $f(u, \mathbf{x}_0) = O(d(\mathbf{x}_0) \cdot h(u))$ as $u \to u_0$ and $d(\mathbf{x}_0) \to 0$, where u is a real variable and $h(u)$ is a function. In order not to repeat the invariant part of this formula, we will use the following abbreviated notation:

$$f(u, \mathbf{x}_0) = \overline{O}(h(u)), \quad \text{as } u \to u_0.$$

We note that for each fixed $\mathbf{x}_0$ this notation is equivalent to the ordinary big-O notation.

In the current proof, the word "constant" will normally mean "a function of $\mathbf{x_0}$", and the dependence on $\mathbf{x}_0$ will not be explicitly reflected in our notation.

Let $y(\tau, \mathbf{y}_0)$ denote the solution to the reduced equation (3.6) with initial condition $\mathbf{y}_0$. Lemma 3.11 applied for $\mathbf{y}_0 = \mathbf{x}_0$ implies that $\mathbf{y}(\tau, \mathbf{y}_0) - \bar{\mathbf{y}} = \overline{O}(\exp(-\rho\tau))$. The function g is Lipschitz, and thus we obtain:

$$g(\mathbf{y}(\tau, \mathbf{y}_0)) - g(\bar{\mathbf{y}}) = \overline{O}(\exp(-\rho\tau)).$$

By (3.21) and Lemma 3.10 we have

$$\begin{aligned} p(\tau) \;&=\; p(0) + \int_0^\tau g(\bar{\mathbf{y}})\,d\tau' + \int_0^\tau (g(\mathbf{y}(\tau', \mathbf{y}_0)) - g(\bar{\mathbf{y}}))\,d\tau'\,. \\ &=\; C_1 + c\tau + \overline{O}(\exp(-\rho\tau)), \end{aligned}$$

where C_1 is a finite constant given by:

$$C_1 \stackrel{\text{def}}{=} p(0) + \int_0^\infty (g(\mathbf{y}(\tau', \mathbf{y}_0)) - g(\bar{\mathbf{y}}))\,d\tau'.$$

Whe note that if $\varphi(\mathbf{x}_0) = 1$ then $p(0) = \ln\varphi(\mathbf{x}_0) = 0$ and

$$C_1 = O(d(\mathbf{x}_0)) = \overline{O}(1).$$

By (3.23) we have

$$\begin{aligned}
\mathbf{x}(\tau) &= \exp(C_1 + c\tau + \overline{O}(\exp(-\rho\tau)))\,\mathbf{y}(\tau) \\
&= \exp(C_1)\,\exp(c\tau)\,(1+\overline{O}(\exp(-\rho\tau)))\,(\bar{\mathbf{y}} + \overline{O}(\exp(-\rho\tau))) \\
&= C_2\,\exp(c\tau)\,\bar{\mathbf{y}} + \overline{O}(\exp((c-\rho)\tau)), \qquad (3.31)
\end{aligned}$$

where $C_2 = \exp(C_1)$. Finally, by (3.22) we obtain

$$\begin{aligned}
t(\tau) &= \int_0^\tau \exp(-(r-1)(C_1 + c\tau' + \overline{O}(\exp(-\rho\tau'))))\,d\tau' \\
&= \int_0^\tau C_3 \exp(-\kappa\tau')(1+\overline{O}(\exp(-\rho\tau')))\,d\tau' \\
&= C_3 \int_0^\tau \exp(-\kappa\tau')\,d\tau' + \overline{O}(\exp(-(\kappa+\rho)\tau')))\,d\tau' \\
&= C_3\bar{t}(\tau) + \overline{O}(\exp(-(\kappa+\rho)\tau')))\,d\tau' \qquad (3.32)
\end{aligned}$$

where $C_3 = C_2^{-(r-1)}$ and the function $\bar{t}$ given by the formula

$$\bar{t}(\tau) = \int_0^\tau \exp(-\kappa\tau')\,d\tau'$$

or explicitly based on formulas (3.24):

$$\bar{t}(\tau) = \begin{cases} \tau & \text{if } \kappa = 0. \\ \dfrac{1}{\kappa}\,(1-\exp(-\kappa\tau)) & \text{if } \kappa \neq 0, \end{cases} \qquad (3.33)$$

We also recall that $\bar{t}$ is 1:1, strictly increasing and $\bar{t}([0,\infty)) = D$, where

$$D = \begin{cases} [0,\infty) & \text{if } \kappa < 0, \\ \left[0, \dfrac{1}{\kappa}\right) & \text{if } \kappa > 0, \\ [0,\infty) & \text{if } \kappa = 0. \end{cases} \qquad (3.34)$$

It will also be useful to consider the inverse of the function $\bar{t}$, the function $\bar{\tau} : D \to [0,\infty)$:

$$\bar{\tau}(t) = \begin{cases} t & \text{if } \kappa = 0, \\ \dfrac{1}{\kappa}\ln\dfrac{1}{1-\kappa t} & \text{if } \kappa \neq 0. \end{cases} \qquad (3.35)$$

Let us prove (1) first, i.e. let us assume that $\kappa = 0$, (which implies that either $r = 1$ or $c = 0$). This is the easiest case, and although it follows from more general estimates that follow, it should help the reader with understanding of the current proof more quickly, before we go into the detailed calculations of the case $\kappa \neq 0$. If $\kappa = 0$ then $\bar{t}(\tau) = \tau$ and $\bar{\tau}(t) = t$, i.e. $t = \tau$ along the dynamic equilibrium ray. In this case, the formula (3.32) reduces to

$$t(\tau) = C_3\tau + \int_0^\tau \overline{O}(\exp(-\rho\tau'))\,d\tau' = C_3\tau + C_4 + \overline{O}(\exp(-\rho\tau))$$

where $C_4 = O(d(\mathbf{x}_0))$ is a constant whose existence follows from Lemma 3.10. This last equation implies

$$\tau = \frac{t - C_4}{C_3} + \overline{O}\left(\exp\left(-\frac{\rho}{C_3}t\right)\right). \qquad (3.36)$$

Let us use the abbreviation

$$\hat{t} = \frac{t - C_4}{C_3}.$$

With this notation:

$$\tau = \hat{t} + \overline{O}\left(\exp\left(-\rho\hat{t}\right)\right).$$

Plugging this equation into (3.31) we obtain

$$\begin{aligned}
\mathbf{x}(t) &= C_2 \exp\left(c\hat{t}\right) \exp\left(\overline{O}\left(\exp\left(-\rho\hat{t}\right)\right)\right) \bar{\mathbf{y}} \\
&\quad + \overline{O}\left(\exp\left((c-\rho)\hat{t}\right)\right) O(\exp(\overline{O}\left(\exp\left(-\rho\hat{t}\right)\right))) \\
&= C_2 \exp\left(c\hat{t}\right) \left(1 + \overline{O}\left(\exp\left(-\rho\hat{t}\right)\right)\right) \bar{\mathbf{y}} + \overline{O}\left(\exp\left((c-\rho)\hat{t}\right)\right) \\
&= C_2 \exp\left(c\hat{t}\right) \bar{\mathbf{y}} + \overline{O}\left(\exp\left((c-\rho)\hat{t}\right)\right) + \overline{O}\left(\exp\left((c-\rho)\hat{t}\right)\right) \\
&= C_2 \exp\left(c\hat{t}\right) \bar{\mathbf{y}} + \overline{O}\left(\exp\left((c-\rho)\hat{t}\right)\right).
\end{aligned}$$

In order to obtain the form in the statement of the theorem, remembering that $C_2 = \exp(C_1)$, we set $c' = \frac{c}{C_3}$, $\rho' = \frac{\rho}{C_3}$, $t_0 = C_4 - \frac{C_1}{c'}$. As we already noted, $C_1 = O(d(\mathbf{x}_0))$. Also, $C_2 = \exp(C_1) = 1 + O(d(\mathbf{x}_0))$, and $C_3 = {C_2}^{-(r-1)} = 1 + O(d(\mathbf{x}_0))$, as $\mathbf{x}_0 \to \bar{\mathbf{y}}$. This concludes the proof of part (1).

Let us now consider the case $\kappa \neq 0$. It proves that the argument presented above applies with small changes to the case when $\kappa > 0$, or even when only $\kappa + \rho > 0$. In this case it is always true that

$$t(\tau) \quad = \quad C_3 \bar{t}(\tau) + C_4 + \overline{O}(\exp(-(\kappa+\rho)\tau). \tag{3.37}$$

where C_4 is a certain constant given by Lemma 3.10. The important distinction between the case $\kappa > 0$ and $\kappa \leq 0$ is that

$$\bar{t}_\infty \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \bar{t}(\tau) = \begin{cases} \infty & \text{if } \kappa \leq 0, \\ \dfrac{1}{\kappa} & \text{if } \kappa > 0. \end{cases}$$

Let us focus first on the case $\kappa < 0$, in which $\bar{t}_\infty$ is infinite. Let

$$\hat{t} = \frac{t - C_4}{C_3}.$$

When $(t, \mathbf{x}_0) \to (\infty, \bar{\mathbf{y}})$, also $(\tau, \mathbf{x}_0) \to (\infty, \bar{\mathbf{y}})$. From (3.37) we obtain:

$$\hat{t} \quad = \quad \bar{t}(\tau) + \overline{O}\left(\exp\left(-\left(\kappa+\rho\right)\tau\right)\right), \tag{3.38}$$

and from (3.36)

$$\tau \quad = \quad \bar{\tau}\left(\hat{t} + \overline{O}(1)\right). \tag{3.39}$$

Let us start with a simple observation that for any constant h:

$$\begin{aligned}
\bar{\tau}(t+h) &= \frac{1}{\kappa} \ln \frac{1}{1-\kappa(t+h)} \\
&= \frac{1}{\kappa} \left(\ln \frac{1}{1-\kappa t} + \ln \frac{1}{1 - \frac{\kappa h}{1-\kappa t}} \right) \\
&= \frac{1}{\kappa} \ln \frac{1}{1-\kappa t} + O\left(\frac{h}{1-\kappa t}\right)
\end{aligned} \tag{3.40}$$

It will also be convenient to introduce the function $\rho : D \to \mathbb{R}$, where D is given by formula (3.34) and ρ is given by the formula:

$$\delta(t) \quad = \quad \frac{1}{1-\kappa t}.$$

It is clear that

$$\exp \bar{\tau}(t) = \delta(t)^{\frac{1}{\kappa}}, \tag{3.41}$$

an identity which will be used many times in what follows.

We can express the asymptotic estimate (3.40) as:

$$\bar{\tau}(t+h) = \bar{\tau}(t) + O(h\delta(t)). \tag{3.42}$$

In addition, for sufficiently small h this estimate is uniform on $[0,\infty)$, i.e. the constant in $O(h\delta(t))$ is uniformly chosen for all $t \in [0,\infty)$ from the fact that $O(h\delta(t))$ is obtained from the Taylor expansion of function $\ln(1+x)$. This last comment is important for obtaining asymptotic estimates which hold throughtout the interval $[0,\infty)$, based on the observation that $C_4 = O(d(\mathbf{x}_0))$ as $\mathbf{x}_0 \to \bar{\mathbf{y}}$. We notice that if this uniformity were not at stake, we could simply write $O(1/t)$ in place of $O(\delta(t))$.

In view of the assumption $\rho+\kappa>0$, equation (3.38) implies $\hat{t} = \bar{t}(\tau) + \overline{O}(1)$ and

$$\tau = \bar{\tau}(\hat{t} + \overline{O}(1)) = \bar{\tau}(\hat{t}) + \overline{O}(1)\delta(\hat{t})) = \bar{\tau}(\hat{t}) + \overline{O}(\delta(\hat{t})). \tag{3.43}$$

This asymptotic equation can be refined in the following way. First, from equations (3.38) and (3.42) we obtain:

$$\begin{aligned} \tau &= \bar{\tau}\left(\hat{t} + \overline{O}(\exp(-(\kappa+\rho)\tau))\right) \\ &= \bar{\tau}\left(\hat{t}\right) + \overline{O}\left(\exp(-(\kappa+\rho)\tau)\delta(\hat{t})\right). \end{aligned}$$

By bootstrapping this equality and using (3.41), (3.43), as $(t,\mathbf{x}_0) \to (\infty,\bar{\mathbf{y}})$, we obtain:

$$\begin{aligned} \tau &= \bar{\tau}(\hat{t}) + \overline{O}\left(\delta(\hat{t})\exp\left(-(\kappa+\rho)\left(\bar{\tau}(\hat{t}) + \overline{O}(\delta(\hat{t}))\right)\right)\right) \\ &= \bar{\tau}(\hat{t}) + \overline{O}\left(\delta(\hat{t})\delta(\hat{t})^{-\frac{\kappa+\rho}{\kappa}}\right) \\ &= \bar{\tau}(\hat{t}) + \overline{O}\left(\delta(\hat{t})^{-\frac{\rho}{\kappa}}\right) \\ &= \bar{\tau}(\hat{t}) + \overline{O}\left(\delta(\hat{t})^{\xi}\right), \end{aligned}$$

where

$$\xi = -\frac{\rho}{\kappa} > 1$$

Plugging this equation into (3.31) we obtain

$$\begin{aligned} \mathbf{x}(t) &= C_2\exp\left(c\left(\bar{\tau}\left(\hat{t}\right) + \overline{O}\left(\delta(\hat{t})^{\xi}\right)\right)\right)\bar{\mathbf{y}} + \overline{O}\left(\exp\left((c-\rho)\left(\bar{\tau}(\hat{t}) + \overline{O}\left(\delta(\hat{t})^{\xi}\right)\right)\right)\right) \\ &= C_2\delta(\hat{t})^{\frac{1}{r-1}}\left(1 + \overline{O}\left(\delta(\hat{t})^{\xi}\right)\right)\bar{\mathbf{y}} + \overline{O}\left(\delta(\hat{t})^{\frac{c-\rho}{\kappa}}\right) \\ &= C_2\delta(\hat{t})^{\frac{1}{r-1}}\bar{\mathbf{y}} + \overline{O}\left(\delta(\hat{t})^{\xi+\frac{1}{r-1}}\right) + \overline{O}\left(\delta(\hat{t})^{\frac{c-\rho}{\kappa}}\right) \\ &= C_2\delta(\hat{t})^{\frac{1}{r-1}}\bar{\mathbf{y}} + \overline{O}\left(\delta(\hat{t})^{\eta}\right) \\ &= \left(\frac{\delta(\hat{t})}{C_3}\right)^{\frac{1}{r-1}}\bar{\mathbf{y}} + \overline{O}\left(\delta(\hat{t})^{\eta}\right) \end{aligned}$$

where

$$\eta = \min\left(\xi + \frac{1}{r-1}, \frac{c-\rho}{\kappa}\right).$$

We note that

$$\frac{c-\rho}{\kappa} = \frac{1}{r-1} - \frac{\rho}{\kappa} = \frac{1}{r-1} + \xi.$$

Thus,

$$\eta = \xi + \frac{1}{r-1} = \frac{c-\rho}{\kappa}.$$

Let us adjust our notation sligthly to bring the result to the form in the statement of the theorem. We note that

$$\begin{aligned}
\frac{\delta(\hat{t})}{C_3} &= \frac{1}{C_3\left(1-\kappa\frac{t-C_4}{C_3}\right)} \\
&= \frac{1}{C_3-\kappa\,(t-C_4)} \\
&= \frac{1}{1-\kappa\left(t-C_4+\frac{C_3-1}{\kappa}\right)} \\
&= \frac{1}{1-\kappa(t-t_0)},
\end{aligned}$$

where $t_0 = C_4 - \frac{C_3-1}{\kappa} = O(d(\mathbf{x}_0))$, as $\mathbf{x}_0 \to \bar{\mathbf{y}}$.

The next case to consider is that of $\kappa > 0$. In this case, formula (3.37) holds and the range of t which corresponds to $\tau \in [0,\infty)$ is $[0, t_\infty)$, where $t_\infty = C_3\bar{t}_\infty + C_4$ and $\bar{t}_\infty = \frac{1}{\kappa}$. Thus, the trajectory has a finite blowup time. Formulas (3.38) and (3.39) hold in this case as well because if $t \nearrow t_\infty$ then also $\tau \to \infty$. For any t within the range $(0, t_\infty)$ the corresponding $\hat{t} = \frac{t-C_4}{C_3}$ is within the range $(0, \bar{t}_\infty)$.

The major difference between the case $\kappa > 0$ and $\kappa < 0$ is that the formula (3.43) cannot be used because $\delta(\hat{t}) \to \infty$ and thus the equation (3.42) would have to be applied with the bound $h = \overline{O}(1)$, which would bring us outside of its range of its validity. Thus, we need to replace formula (3.43) with a more precise estimate. The idea is to go back to equation (3.38) and use it to compare the rates of approach of $\hat{t}$ and $\bar{t}(\tau)$ towards $\bar{t}_\infty = \frac{1}{\kappa}$. We obtain:

$$\begin{aligned}
\frac{1}{\kappa} - \hat{t} &= \left(\frac{1}{\kappa} - \bar{t}(\tau)\right) + \overline{O}(\exp(-(\kappa+\rho)\tau)) \\
&= \frac{1}{\kappa}\exp(-\kappa\tau) + \overline{O}(\exp(-(\kappa+\rho)\tau)) \\
&= \exp(-\kappa\tau)\left(\frac{1}{\kappa} + \overline{O}(\exp(-\rho\tau)\right) \\
&= \exp(-\kappa\tau)\left(\frac{1}{\kappa} + \overline{O}(1)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\ln(\bar{t}_\infty - \hat{t}) &= -\kappa\tau + \ln\left(\frac{1}{\kappa} + \overline{O}(1)\right) \\
&= -\kappa\tau + \overline{O}(1).
\end{aligned}$$

Therefore, as $(t, \mathbf{x}_0) \to (\bar{t}_\infty, \bar{\mathbf{y}})$, or, equivalently, $(\hat{t}, \mathbf{x}_0) \to (\bar{t}_\infty, \bar{\mathbf{y}})$,

$$\tau = \frac{1}{\kappa}\ln\frac{1}{\bar{t}_\infty - \hat{t}} + \overline{O}(1).$$

This equation provides the sufficient quantitative information about the growth of $\tau \to \infty$ in order to proceed with the bootstrapping argument similar to the case $\kappa < 0$. Indeed,

$$\begin{aligned}
\tau &= \bar{\tau}\left(\hat{t} + \overline{O}(\exp(-(\kappa+\rho)\tau))\right) \\
&= \bar{\tau}\left(\hat{t} + \overline{O}\left(\exp\left(-(\kappa+\rho)\left(\frac{1}{\kappa}\ln\frac{1}{\bar{t}_\infty - \hat{t}} + \overline{O}(1)\right)\right)\right)\right) \\
&= \bar{\tau}\left(\hat{t} + \overline{O}\left(\bar{t}_\infty - \hat{t}\right)^{1+\frac{\rho}{\kappa}}\right) \\
&= \bar{\tau}(\hat{t}) + \overline{O}\left(\left(\bar{t}_\infty - \hat{t}\right)^{1+\frac{\rho}{\kappa}}\delta(\hat{t})\right) \\
&= \bar{\tau}(\hat{t}) + \overline{O}\left(\bar{t}_\infty - \hat{t}\right)^{\frac{\rho}{\kappa}}.
\end{aligned}$$

We used the fact that $\delta(\hat{t}) = O(\bar{t}_\infty - \hat{t})^{-1}$, and although it grows as $t \to \bar{t}_\infty$, its growth is defeated by the power $1 + \frac{\rho}{\kappa}$ of the other term. Also, we can see easily that we applied equation (3.42) within its region of validity. The remainder of the argument is essentially identical to the case $\kappa < 0$.

Our last remaining claim is that the first term of the right hand side of formula (3.30) is dominant. We note that

$$\frac{\|(\text{second term})\|}{\|(\text{first term})\|} = \begin{cases} \overline{O}(\exp(-\rho'(t-t_0))) & \text{if } \kappa = 0, \\ \overline{O}(1-\kappa(t-t_0))^{\frac{\rho}{\kappa}} & \text{if } \kappa \neq 0. \end{cases} \tag{3.44}$$

Let us apply a proof by cases:

(1) When $\kappa = 0$, clearly the ratio goes to 0 as $(t, \mathbf{x}_0) \to (\infty, \bar{\mathbf{y}})$.
(2) When $\kappa \neq 0$ we have two subcases:
  (a) If $\kappa < 0$ then $t \to \infty$ and thus $1 - \kappa t \to \infty$. The power $\frac{\rho}{\kappa} < 0$, and thus the ratio also goes to 0 as $(t, \mathbf{x}_0) \to (\infty, \bar{\mathbf{y}})$.
  (b) If $\kappa > 0$ then $1 - \kappa t \to 0$, as $t \nearrow t_0 + \frac{1}{\kappa}$. However, in this case the power $\frac{\rho}{\kappa} > 0$, and thus the ratio also goes to 0 as $t \nearrow t_0 + \frac{1}{\kappa}$ and $\mathbf{x}_0 \to \bar{\mathbf{y}}$.

Our proof is complete now. ☐

Corollary 3.13. *Let us assume that $\bar{\mathbf{y}}$ is a normally linearly dynamically stable dynamic equilibrium. Equivalently, let for some real number c $\mathbf{F}(\bar{\mathbf{y}}) = c\bar{\mathbf{y}}$ and let the spectrum $\sigma(D\mathbf{F}(\mathbf{y}))$ admit a decomposition into a union of a simple eigenvalue r c and a closed set σ_0 such that*

$$\sup_{z\in\sigma_0} \Re z < \min(c, rc).$$

Then the ray $\mathscr{R}(\bar{\mathbf{y}})$ of dynamic equilibria is an attractor for any trajectory with initial condition chosen from an open cone $W \supset \mathscr{R}(\bar{\mathbf{y}})$ such that the trajectory is attracted exponentially or according to a power law along the direction of the ray to either **0** *or ∞, or to a fixed point on the ray, depending on the sign of c. In more detail, if $c < 0$, then the trajectory is attracted to the ray with an exponential speed (when $r = 1$) or a power speed (when $r \neq 1$) as it converges exponentially (when $r = 1$) or according to a power law (when $r \neq 1$) to* **0** *along the direction of the ray; if $c > 0$, then the trajectory is attracted to the ray with an exponential speed (when $r = 1$) or a power speed (when $r \neq 1$) as it blows up exponentially (when $r = 1$) or in accordance with a power law (when $r \neq 1$) to ∞ along the direction of the ray; if $c = 0$, then the trajectory is attracted to a fixed point on the ray with exponential speed.*

Proof. Since $c - \rho < 0$, when $\kappa < 0$, then $\eta = \frac{c-\rho}{\kappa} > 0$; when $\kappa > 0$, then $\eta = \frac{c-\rho}{\kappa} < 0$. When $\kappa = 0$, we have $c' - \rho' = (c-\rho)/C_3 < 0$ because $C_3 = \exp(C_1)^{-(r-1)} > 0$. Thus in all cases, the second term of the asymptotic expansion is an exponential or power decay function. Therefore the proof is complete. ☐

4. THE DYNAMIC MODEL AND STABILITY ANALYSIS

In this section we will apply our abstract results concerning dynamic equilibria in homogenous systems to our economic model. Our analysis will be applicable both to the simplified model (1.7) and to the corresponding infinite-dimensional model. Let $X = \mathbb{R}^n$ or $X = \ell^\infty$, and $U = \{\mathbf{y} : y_i > 0\}$ in the finite-dimensional case, and $U = \{\mathbf{y} : M(\mathbf{y}) < \infty \text{ and } m(\mathbf{y}) > 0\}$ in the infinite-dimensional case. Let us consider the vector field $\mathbf{F} : U \subset X$ defined by the formulae (used before, but gathered here for the convenience of the reader):

$$\begin{aligned}
\lambda &= (\lambda_1, \lambda_2, \ldots) \in \ell^\infty \\
s(\mathbf{y}) &= \sum_i \gamma_i y_i \\
\mathbf{F}_0(\mathbf{y}) &= \left(\frac{\lambda_1}{y_1^2}, \frac{\lambda_2}{y_2^2}, \ldots\right) \\
\mathbf{F}(\mathbf{y}) &= -\frac{1}{s(\mathbf{y})^2}\lambda + \mathbf{F}_0(\mathbf{y})
\end{aligned}$$

We note that under the standard identification $(\mathbb{R}^n)^* = \mathbb{R}^n$ or $(\ell^\infty)^* = \ell^1$,

$$s = (\gamma_1, \gamma_2, \ldots).$$

We note that for every vector $\mathbf{h} \in X$ we have:

$$D\mathbf{F}(\mathbf{y})\mathbf{h} = \frac{2}{s(\mathbf{y})^3} s(\mathbf{h})\lambda + D\mathbf{F}_0(\mathbf{y})\mathbf{h}.$$

Also, it is essential that $D\mathbf{F}_0(\mathbf{y})$ is a diagonal operator, i.e. the standard basis of X or ℓ^∞ consists of eigenvectors of $D\mathbf{F}_0(\mathbf{y})$. Indeed,

$$D\mathbf{F}_0(\mathbf{y})\mathbf{h} = \left(-\frac{2\lambda_1}{y_1^3} h_1, -\frac{2\lambda_2}{y_2^3} h_2, \ldots\right).$$

and thus for $i = 1, 2, \ldots$:

$$D\mathbf{F}_0(\mathbf{y})\mathbf{e}_i = -\frac{2\lambda_i}{y_i^3}\mathbf{e}_i.$$

where $\mathbf{e}_i = (\delta_{ij})$. (We note that the span of this set is not even dense in ℓ^∞!) Symbolically,

$$D\mathbf{F}_0(\mathbf{y}) = \operatorname{diag}\left(-\frac{2\lambda_i}{y_i^3}\right).$$

We note that if $\inf_i \lambda_i > 0$ and $\sup_i \lambda_i < \infty$, as we will assume, $\mathbf{F}_0 : U \to U$ is a *diffeomorphism*, in particular, $D\mathbf{F}_0(\mathbf{y})$ is a *linear isomorphism*. The operator $\mathbf{e} \otimes s$ is a rank 1 operator, and thus it is a *compact* linear operator. A sum of an isomorphism and a compact operator is called a *Fredholm operator*, and there are plenty of useful facts about operators in this class. For instance, the continuous part of the spectrum of the operators $D\mathbf{F}(\mathbf{y})$ and $D\mathbf{F}_0(\mathbf{y})$ coincide, and thus both are contained in the spectrum of the latter:

$$\sigma(D\mathbf{F}_0(\mathbf{y})) = \overline{\left\{-\frac{2\lambda_i}{y_i^3} : i = 1, 2, \ldots\right\}},$$

(here $\overline{A}$ denotes the closure of the set A) while the operator $D\mathbf{F}(\mathbf{y})$ could have some additional discrete spectrum, consisting of isolated poles of its resolvent. Of course, these comments are only relevant in the infinite-dimensional case.

The above comments can be made explicit by calculating the resolvents. We have,

$$R(z, D\mathbf{F}_0(\mathbf{y})) = (z\mathbf{I} - D\mathbf{F}_0(\mathbf{y}))^{-1} = \operatorname{diag}\left(\left(z + \frac{2\lambda_i}{y_i^3}\right)^{-1}\right).$$

The resolvent of a sum of operators is calculated using the *resolvent formula*:

$$R(z,A+B)=\sum_{k=0}^{\infty}R(z,A)(BR(z,A))^k=\sum_{k=0}^{\infty}(R(z,A)B)^kR(z,A).$$

Proof. We have

$$\begin{aligned}(zI-A-B)^{-1} &= ((zI-A)-B)^{-1}=R(z,A)(I-BR(z,A))^{-1}\\ &= \sum_{k=0}^{\infty}R(z,A)(BR(z,A))^k.\end{aligned}$$

□

This formula works particularly well if B is a rank 1 operator, i.e. $B=a\otimes\mu$ where $a\in X$ and $\mu\in X^*$. We recall that the tensor product of a vector and a functional is defined as follows: for every $u\in X$

$$Bu=\mu(u)a.$$

If $R=R(z,A)$ then $BR=a\otimes R^*(\mu)$, $RBR=R(a)\otimes R^*(\mu)$, $BRBR=\mu(R(a))a\otimes R^*(\mu)$, $RBRBR=\mu(R(a))R(a)\otimes R^*(\mu)$, $BRBRBR=\mu(R(a))^2a\otimes R^*(\mu)$, $RBRBRBR=\mu(R(a))^2R(a)\otimes R^*(\mu)$, etc. and thus:

$$R(z,A+B)=R+\sum_{k=0}^{\infty}\mu(R(A))^kR(a)\otimes R^*(\mu)=R+(1-\mu(R(a))^{-1}R(a)\otimes R^*(\mu).$$

Hence, the resolvent of $A+B$ is a rank 1 perturbation of the resolvent of A.

The resolvent formula, applied to our situation yields:

$$R(z,D\mathbf{F}(\mathbf{y}))=R(z,D\mathbf{F}_0(\mathbf{y}))+f(z)^{-1}p(z)\otimes q(z)$$

where, using $a=\lambda$, $\mu=\frac{2}{s(\mathbf{y})^3}s$ and $R=R(z,D\mathbf{F}_0(\mathbf{y}))$:

$$\begin{aligned}f(z) &= 1-\frac{1}{s^3}s(R\lambda)=1-\frac{2}{s^3}\sum_i\frac{\gamma_i\lambda_i}{z+\frac{2\lambda_i}{y_i^3}},\\ p(z) &= R\lambda=\left(\frac{\lambda_i}{z+\frac{2\lambda_i}{y_i^3}}\right)\in\ell^\infty,\\ q(z) &= R^*\mu=\left(\frac{2\gamma_i}{s^3}\frac{1}{z+\frac{2\lambda_i}{y_i^3}}\right)\in\ell^1.\end{aligned}$$

Thus, we can see directly that $R(z,D\mathbf{F}(\mathbf{y}))$ is meromorphic in the complement of $\sigma(D\mathbf{F}_0(\mathbf{y}))$ and it has poles wherever $f(z)=0$.

It is also easy to see that if $\Im z>0$ then $\Im f(z)>0$. Indeed,

$$\begin{aligned}\Im f(z) &= -\frac{1}{s^3}\sum_i\gamma_i\Im\frac{1}{z+\frac{2\lambda_i}{y_i^3}}\\ &= -\frac{1}{s^3}\sum_i\gamma_i\Im\frac{\bar z+\frac{2\lambda_i}{y_i^3}}{|z+\frac{2\lambda_i}{y_i^3}|^2}\\ &= -\frac{1}{s^3}\sum_i\gamma_i\frac{-\Im z}{|z+\frac{2\lambda_i}{y_i^3}|^2}>0.\end{aligned}$$

Similarly if $\Im z<0$ then $\Im f(z)<0$. Thus, all roots of $f(z)$ are on the real axis. We also notice that on the real axis $f'(z)>0$ and thus $f(z)$ is strictly decreasing. Moreover, if $(z_1,z_2)\subset\mathbb{R}$ is a maximal finite interval in the complement of $\sigma(D\mathbf{F}_0(\mathbf{y}))$ then the function $f(z)$ varies on that interval from $-\infty$ at z_1 to $+\infty$ at z_2. Hence, in each such interval there is one simple root of $f(z)$. Moreover, on a maximal interval $(-\infty, z_2)$ this function is > 1 and thus it has no roots. In addition, on the maximal interval (z_1, ∞) the function varies from $-\infty$ to 1 and thus has an additional simple root.

Below, we summarize earlier results, adapting them to the notations of this section, and covering the infinite-dimensional case as well.

Let us consider dynamic equilibria, i.e. solutions of the equation $\mathbf{F}(\mathbf{y}) = c\mathbf{y}$. We already noticed that if $\mathbf{F}(\theta\mathbf{y}) = \theta^r\mathbf{F}(\mathbf{y})$ for all $\theta > 0$ then $D\mathbf{F}(\mathbf{y})\mathbf{y} = r\,c\,\mathbf{y}$ (Euler's identity). Thus, $r\,c$ is an eigenvalue of $D\mathbf{F}(\mathbf{y})$ with the eigenvector $\mathbf{y}$. In our situation, $r = -2$. The dynamic equilibrium equation can be also written as:

$$\lambda_i\left(-\frac{1}{s^2}+\frac{1}{y_i^2}\right) = cy_i, \quad i = 1, 2\ldots$$

We observed that homogeneity allows us to assume $s = 1$. Thus, we consider a system of equations:

$$\left\{\begin{array}{rcl} \lambda_i\left(-\frac{1}{y_i}+\frac{1}{y_i^3}\right) & = & c, \\ \sum_i \gamma_i y_i & = & 1. \end{array}\right. \tag{4.1}$$

We already noticed that the left-hand sides of the first equation have positive minima at $y_i = \sqrt{3}$, with the minimum values of $-\frac{2\lambda_i}{\sqrt{3}}$. Thus, there are no values of *expansion coefficient* c such that $c < -\frac{2\inf_i \lambda_i}{\sqrt{3}}$. Moreover, if $\sum_i \gamma_i = 1$ then $c = 0$ is the only possible value of the *expansion coefficient* and $\mathbf{y} = (1, 1, \ldots)$. (this case we have studied in detail). If $\sum_i \gamma_i > 1$ then $c > 0$ and we have a unique dynamic equilibrium.

Let us note that $-2c + \frac{2\lambda_i}{y_i^3} = \frac{2\lambda_i}{y_i} > 0$. Hence,

$$f(-2c) = 1 - \sum_i \frac{2\gamma_i\lambda_i}{\frac{2\lambda_i}{y_i}} = 1 - \sum_i \gamma_i y_i = 0$$

as expected. Moreover, $-2c$ is to the right of $\sigma(R(z, D\mathbf{F}_0(\mathbf{y})))$. Due to our discussion of the zeros of $f(z)$, $-2c$ is the largest eigenvalue of the operator $D\mathbf{F}(\mathbf{y})$, and thus we obtain the following result:

Theorem 4.1. *Let us assume* $\sup_i \lambda_i < \infty$ *and* $\inf_i \lambda_i > 0$. *Every dynamic equilibrium of the system*

$$\dot{y}_i = \lambda_i\left(-\frac{1}{s^2}+\frac{1}{y_i^2}\right)$$

is linearly dynamically stable. Moreover, if c is the expansion coefficient then $-2c$ is the largest eigenvalue of the linearized system at a dynamic equilibrium, while all remaining eigenvalues are real and not greater than

$$\sup_i \frac{-2\lambda_i}{y_i^3} \le -\frac{2\inf_i \lambda_i}{m(\mathbf{y})^3}$$

and are all uniformly $< -2c - \rho_0$, *where*

$$\rho_0 = \inf_i \frac{2\lambda_i}{y_i} > 0.$$

Remark 4.2. The fact that $-2c$ is the largest eigenvalue implies that the ray of dynamic equilibria is a *normally hyperbolic invariant manifold* for the system. This implies the existence of an invariant foliation of its cone neighborhood. In [4] normal hyperbolicity was used to prove the existence of the stable foliation in the case of an ordinary equilibrium. The results can be generalized to the case of a dynamic equilibrium as well. The practical significance of normal hyperbolicity is a further improvement of the asymptotic estimates given in the previous section. Also, normal hyperbolicity results in the tails of the asymptotic formulas in Theorem 3.12 to go to 0 in the norm.

4.1. **A perturbation result.** It is natural to ask whether dynamic equilibria in our system are dynamically stable. The first result concerns small perturbations of a system with $\sum_i \gamma_i = 1$. We will assume that the vector $\gamma = (\gamma_1, \gamma_2, \ldots)$ with $\sum_i \gamma_i$ is replaced with a proportional vector $(1+\epsilon)\gamma$, and thus our perturbation parameter is ϵ. For $\epsilon = 0$ the eigenvalue 0 is a simple pole of the resolvent of the linearized system, while other eigenvalues are negative. Without difficulty, we can see that *cr* remains a simple pole of the resolvent under small perturbations, while all other eigenvalues remain negative. Thus, our system admits stable dynamic equilibria with both $c > 0$ and $c < 0$.

5. Further generalization

We considered the system (1.5. for a finite and infinite set of indices. However, a futher generalization can be considered, when we want to discuss a very large number of firms, so that it makes sense to speak of the distribution of the production vector.

Thus, let (Ω, Σ, μ) be a measure space, where ω represents a firm (or a "randomly chosen" firm). Let $x(\omega)$ represents the output of the firm ω. The output of the industry is measured now by

$$s = \int_\Omega x(\omega)\, d\mu(\omega) = \mathbb{E}\, x,$$

i.e. it is the expected value of the random variable x. Of course, in order to take the expectation we must assume that the function x is Σ-measurable and Lebésgue-integrable.

Let $a : \Omega \to \mathbb{R}$, $\beta : \Omega \to \mathbb{R}$, $k : \Omega \to \mathbb{R}$ and $\gamma : \Omega \to \mathbb{R}$ be positive, measurable functions. We can formulate a system analogous to (1.5) as follows. Let us consider a varying x, i.e. $x : \mathbb{R} \times \Omega \to \mathbb{R}$, or $x(t, \omega)$. Let us form the differential equation:

$$\frac{dx(t,\omega)}{dt} = k(\omega) \left(-\frac{b}{a(\omega)\, s(t)^2} + \frac{\beta(\omega)}{a(\omega)\, x(t,\omega)^2} \right) \tag{5.1}$$

where

$$s(t) = \int_\Omega x(\omega, t)\, d\mu(\omega).$$

We note that we can think of this system as obtained by formally substituting the index i in (1.5) with ω. The only difference is that now we require that the set of "indices" Ω be endowed with a σ-algebra and a measure. We also note that both the finite and countable cases are special cases of our generalized problem, when the counting measure on the set of indices serves as μ.

The above equation is a stochastic differential equation which from the point of view of the individual firm is the same as in our finite and countable version, i.e. it expresses the desire of the individual firm to maximize profit per unit of labor. The only difference is the way the output of the industry s is computed.

Most of our results in the finite and countable case were obtained using a change of variables, which in the new situation can be expressed as:

$$y(\omega) = \frac{\sqrt{b}}{\sqrt{\beta(\omega)}} x(\omega).$$

Let $\lambda : \Omega \to \mathbb{R}$ and $\gamma : \Omega \to \mathbb{R}$ be defined by

$$\begin{aligned} \lambda(\omega) &= \frac{\sqrt{b}}{\sqrt{\beta(\omega)}} \frac{k(\omega) b}{a(\omega)}, \\ \gamma(\omega) &= \frac{\sqrt{\beta(\omega)}}{\sqrt{b}}, \end{aligned}$$

in analogy with the finite and countable cases. This change of variables leads to the simplified stochastic differential equation:

$$\frac{dy(t,\omega)}{dt} = \lambda(\omega)\left(-\frac{1}{s(t)^2} + \frac{1}{y(t,\omega)^2}\right). \tag{5.2}$$

where now

$$s(t) = \int_\Omega y(t,\omega)\gamma(\omega)\,d\mu(\omega) = \int_\Omega y(t,\omega)\,d\nu(\omega).$$

and ν is the measure $\gamma \cdot \mu$. Also, we will assume that γ is integrable, by analogy with the summability condition on γ_i that we needed in the countable case. This is the same as assuming that the measure ν is finite. In addition, there are three distinct cases: $\nu(\Omega) < 1$ (total output shrinking), $\nu(\Omega) = 1$ (total output constant) and $\nu(\Omega) > 1$ (total output increasing).

As usual, we will require that $y \in L^\infty(\Omega, \Sigma, \mu)$, i.e. that y is bounded by a uniform constant except for a set of measure 0. Also, functions

$$\begin{aligned} M(y) &= \sup_{\omega\in\Omega}\operatorname{ess} y(\omega), \\ m(y) &= \inf_{\omega\in\Omega}\operatorname{ess} y(\omega), \end{aligned}$$

play the same role, i.e. they are decreasing and increasing along the trajectories of our system.

With these definitions, our stability results can be generalized to the new system. For instance, if

$$0 < \inf_{\omega\in\Omega}\operatorname{ess}\lambda \le \sup_{\omega\in\Omega}\operatorname{ess}\lambda < \infty$$

then our system goes exponentially fast (in norm) to an equilibrium which is a constant function.

6. Some final remarks

In the current paper we were able to fully develop a theory of Nash equilibria for a specific economic model. In the course of our analysis we investigated the class of homogenous systems. In particular, in order to eliminate a particular economic restriction that the economy never expands or contracts, we defined the notion of a *dynamic equilibrium*. We also defined the notion of stability and linear stability for dynamic equilibria. We examined several techniques which prove the existence of dynamic equilibria, and analyzed their stability.

Throughout the paper we considered both finite- and infinite-dimensional systems of differential equations. We showed that many results also hold for the infinite-dimensional model of competition analogous to the finite-dimensional economic model.

In our paper very few properties were specific to the system that we have studied. In abstract terms, the class of systems to which some or all of the results in this paper apply is the class of systems of differential equations of the form

$$\dot{y}_i = f_i(y_i) - f_i(s), \qquad i = 1, 2, \ldots,$$

where $f_i(y_i)$ is a convex function defined for $y_i > 0$ for every i, and s is some weighted average of the coordinates: $s = \sum_i \gamma_i y_i$, where $\gamma_i > 0$ and $\sum_i \gamma_i < \infty$. In the case $\sum_i \gamma_i = 1$ we can expect a unique stable equilibrium, in the case $\sum_i \gamma_i > 1$ there should be a unique dynamic equilibrium with the *expansion coefficient* positive, while for $\sum_i \gamma_i < 1$ many dynamic equilibria with negative expansion coefficients can be expected.

The results of this paper require only very minor modifications to cover all power functions $f_i(y_i) = \lambda_i y_i^r$. In the paper $r = -2 < 0$, but this condition is not necessary: any $r < 0$ or $r \ge 1$ works as well. The hardest assumption to satisfy is that of homogeneity. For functions of one variable only power functions occur. However, if y_i is allowed to be a vector, many more examples are available.

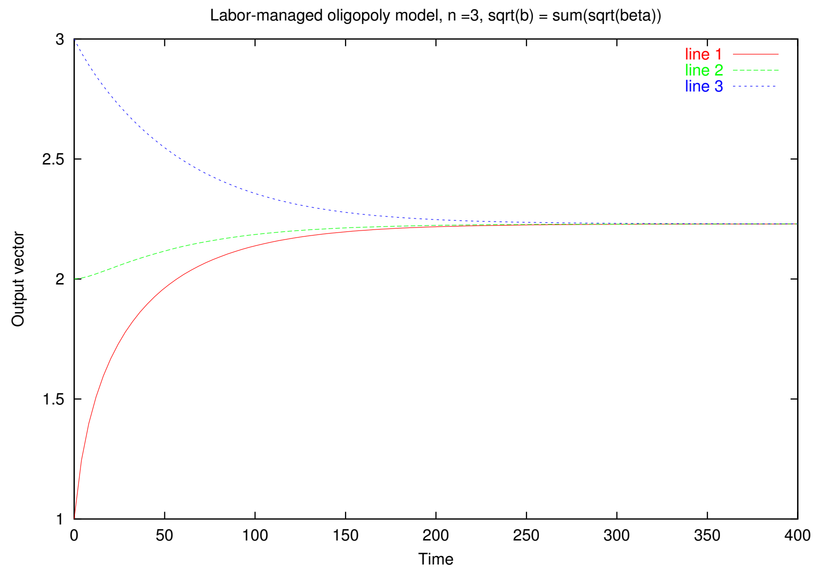

FIGURE 7.1. Output vector when an equilibrium exists. More precisely, we used $k_j = 1$, $\beta_j = 1/9$, $a_j = 1$ and $b = 1$.

7. NUMERICAL EXAMPLES

This short section contains some numerically calculated figures: Figure 7.1, Figure 7.2 and Figure 7.3, illustrating the behavior of the output vector for model (1.5) with $n = 3$. The main point is that the dynamic equilibrium will actually be observed despite the unlimited growth of the output vector. Indeed, in the model $r = -2$ and thus the output vector grows like $t^{\frac{1}{3}}$ (cf. Figure 7.2). This growth is very slow as comparted to the exponential rate of convergence of quantities like x_j/s to their limiting values.

The above comments relate to the issue of practical usefulness of the model and its dynamic equilibria. As our model allows unlimited growth, it cannot be correct "forever". However, the same can be said about most models of temporal behavior. The value of a model based on dynamic equilibria lies in its ability to handle two dramatically different time scales. On one scale, the economy is growing according to a power law. On another time scale, the market share of individual firms converges exponentially to a limit. It is our contention that both time scales are important.

We note that we used Octave as the numerical tool, under the Linux operating system on an Intel Pentium 4 processor. In particular, the functions *lsode* was used, with the default values of its options, resulting in a pretty good stiff integrator. In all examples we used parameter settings identical for all coordinates, which results in the equilibria (dynamic or standard) to be proportional to the vector $(1/3, 1/3, 1/3)$. The convergence to the equilibrium can be seen as the convergence of the three coordinates of the output vector. The reader who repeats our numerical experiments will observe the breakdown of the algorithm for the third example (cf. Figure 7.3). We stopped our integration at $t = 330$, as for higher values the adaptive step would go to 0, and the algorithm would "recover" by letting the values of x_j jump into the negative domain.

In our final experiment, we use a larger number of firms ($n = 10$). We set $a_j = k_j = 1$, but $\beta_j = 1/j$ and $b = .9(\sum_{j=1}^{n} \sqrt{\beta_j})^2$, so that the economy expands. As the pure output vector vs. time plot is somewhat difficult to interpret (cf. Figure 7.4), we also plotted the market share vs. time (Figure 7.5). The experiment clearly demonstrates that market share tends to a constant value. In this example, the limit value of the market share depends on j.

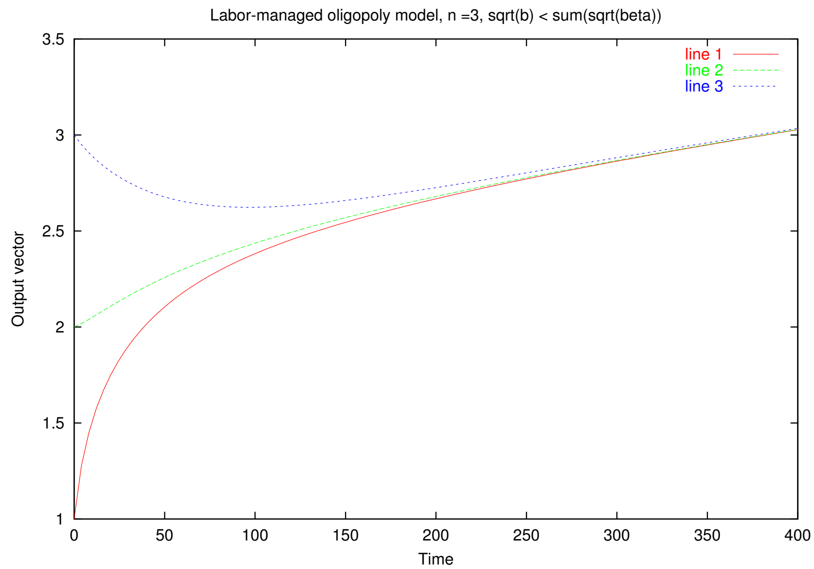

FIGURE 7.2. Output vector when a dynamic equilibrium exists and the economy expands. More precisely, we used $k_j = 1$, $\beta_j = 1/8$, $a_j = 1$ and $b = 1$.

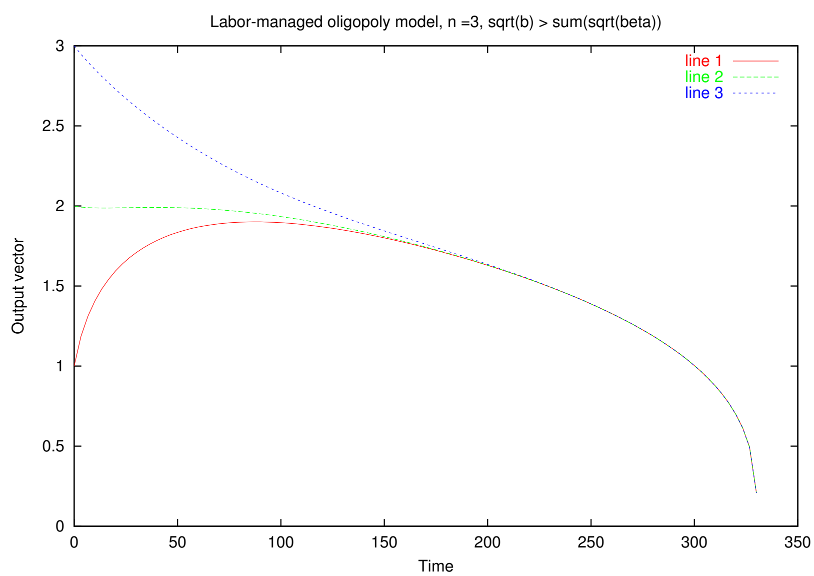

FIGURE 7.3. Output vector when a dynamic equilibrium exists and the economy shrinks. More precisely, we used $k_j = 1$, $\beta_j = 1/10$, $a_j = 1$ and $b = 1$.

APPENDIX A. BANACH SPACES OF SUMMABLE SEQUENCES

In the current paper we choose to work with ℓ^p-norms on the space of either finite sequences $\mathbb{R}^n$ or infinite sequences (a subset of $\mathbb{R}^\infty$).

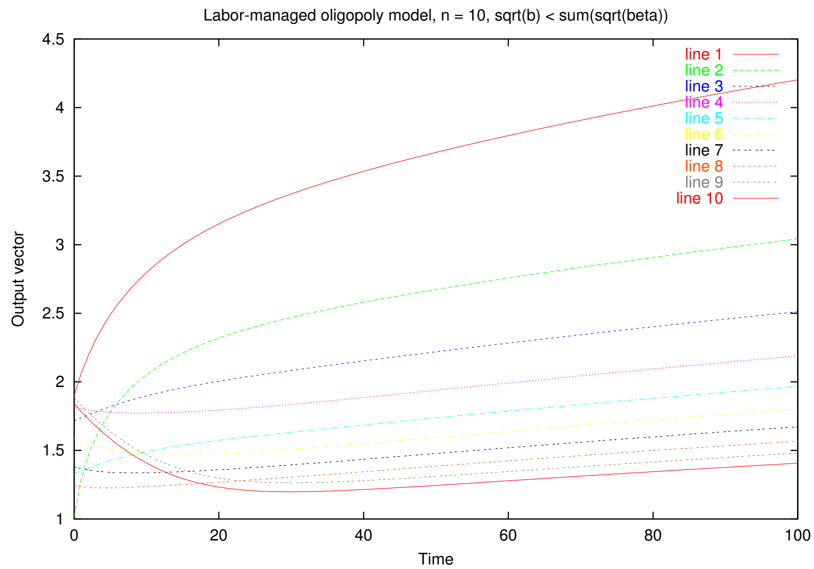

FIGURE 7.4. An example of a higher number of firms, $n = 10$.

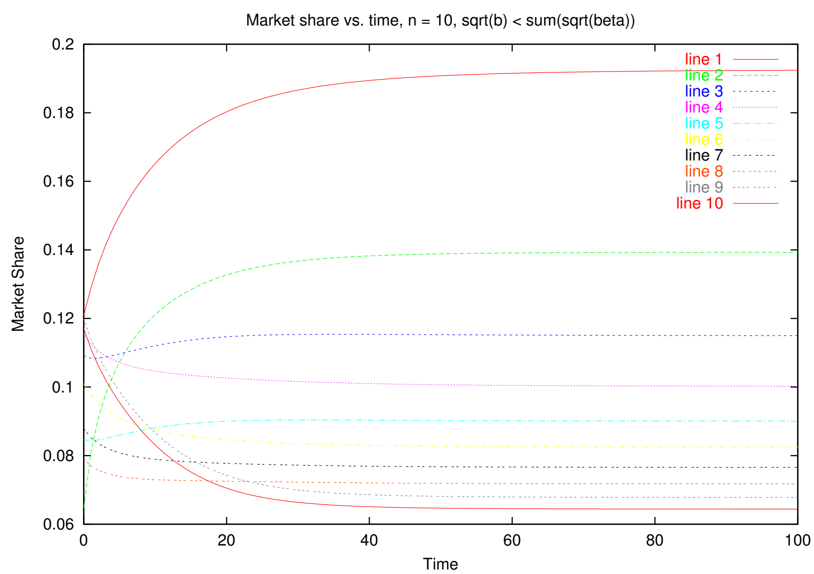

FIGURE 7.5. Market share as function of time, $n = 10$.

Let us summarize basic facts about these norms. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the finite case and $\mathbf{x} = (x_1, x_2, \ldots)$ in the infinite case. In both cases, for any $p \geq 1$:

$$\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}.$$

In the infinte case, the space of sequences for which $\|\mathbf{x}\|_p < \infty$ is called ℓ^p. In most cases, our results will be equally applicable to $\mathbb{R}^n$ with the ℓ^p norm and to the ℓ^p space of infinite sequences.

We recall that $p = \infty$ is a "limit case" as $p \to \infty$, and

$$\|\mathbf{x}\|_\infty = \sup_i |x_i|.$$

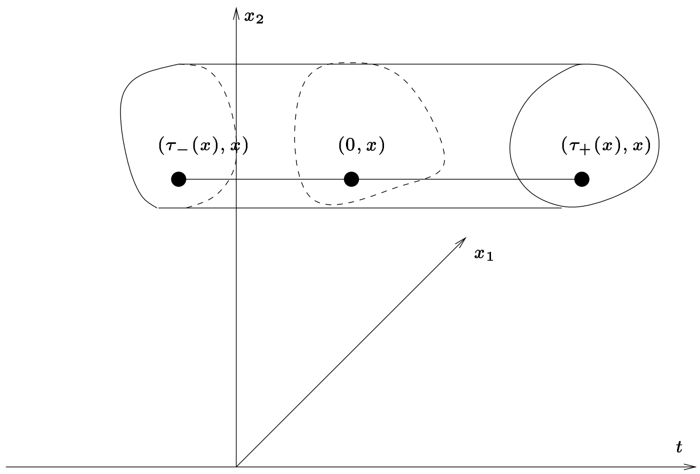

FIGURE B.1. The domain of the local flow.

This norm is also called the sup-norm.

It is well-known that ℓ^p is a Banach space (see, for example, [**?**]).

The dual space to ℓ^p is ℓ^q, where q is computed from the formula:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We will need the case of $p = 0$ and $p = \infty$, i.e. $(\ell^\infty)^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$. The equality is understood by using standard pairing between ℓ^p and ℓ^q: if $\mathbf{x} \in \ell^p$ and $\mathbf{y} \in \ell^q$ then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$$

is finite by Hölder inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

The identification of ℓ^p with $(\ell^q)^*$ is performed by the map:

$$\ell^p \ni \mathbf{x} \mapsto \varphi_{\mathbf{x}} \in (\ell^q)^*$$

where $\varphi_{\mathbf{x}}$ is the map

$$\mathbf{y} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle.$$

APPENDIX B. LOCAL FLOW INDUCED BY A DIFFERENTIAL EQUATION

We note that the Existence and Uniqueness Theorem of differential equations is a *local* theorem. Let X be a Banach space and let $U \subseteq X$ be an open subset. We assume that $\mathbf{F} \in C^1(U, X)$. Let us consider a differential equation on U:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{B.1}$$

We refer to $\mathbf{F}$ as the *vector field* associated with this differential equation.[4] The *local flow* of the system (B.1) is the tripple $(\tau_-, \tau_+, \varphi)$ consisting of:

[4]As we consider vector fields on open subsets of Banach spaces, the distinction between functions mappings $\mathbf{F} \in C^1(U, X)$ and vector fields is not essential. However, vector fields *are not mappings* from U to V but sections of a tangent bundle. This distinction becomes important when developing the theory of differential equations on manifolds, e.g. as it is done in [2].

(1) Two functions $\tau_+ : U \to \mathbb{R} \cup \{\infty\}$ and $\tau_- : U \to \mathbb{R} \cup \{-\infty\}$. The function τ_+ is strictly positive and upper-semicontinuous. The function τ_- is strictly negative and lower-semicontinuous. These functions determine an open subset (cf. Figure B.1):
$$D = D(\tau_-, \tau_+) = \{(t, \mathbf{x}) \in D \subseteq \mathbb{R} \times U \ : \ \tau_-(\mathbf{x}) < t < \tau_+(\mathbf{x})\}.$$
(the openness follows from semi-continuity);

(2) A map $\varphi : D \to M$ such that for any fixed $\mathbf{x}_0 \in U$ the path
$$(\tau_-(\mathbf{x}), \tau_+(\mathbf{x})) \ni t \mapsto \varphi(t, \mathbf{x}_0) \in U$$
is an integral curve of (B.1) passing through $\mathbf{x}_0$ and defined on a *maximal interval*.

The notation $\varphi^t(\mathbf{x})$ is often used in place of $\varphi(t, \mathbf{x})$. The restricted *group property* follows from these definitions: if $t \in (\tau_-(\mathbf{x}), \tau_+(\mathbf{x}))$ and $s \in \left(\tau_-(\varphi^t(\mathbf{x})), \tau_+(\varphi^t(\mathbf{x}))\right)$ then $s + t \in \left(\tau_-(\varphi^{s+t}(\mathbf{x})), \tau_+(\varphi^{s+t}(\mathbf{x}))\right)$ and:
$$\varphi^{s+t}(\mathbf{x}) = \varphi^s(\varphi^t(\mathbf{x}))$$
In many textbooks it is often assumed in formulations of important theorems that $\varphi^t : (-\infty, \infty) \to U$ holds, i.e. that the system is *complete*. However, in this paper the systems under consideration are typically not complete. For a further discussion of completeness, see the appendices in [7].

When dealing with differential equations on Banach spaces which result from partial differential equations, for example, Naviér-Stokes equation, we often can only prove the local existence for $t \geq 0$. In this situation, $\tau_- \equiv 0$ and we speak of a *local semi-flow* instead. There is one more difference: $D = \{(t, \mathbf{x}) \in D \subseteq \mathbb{R} \times U \ : \ 0 \leq t < \tau_+(\mathbf{x})\}$. In particular, D is no longer an open set. Also, $\phi(t, \mathbf{x})$ needs only to be *right-differentiable* at $t = 0$.

Often one discusses semi-flows only eventhough the system can be integrated backwards for a short period of time, and then it blows up to ∞ or goes to the boundary of U. In these situations, the discussion of backward solutions is often of secondary importance.

APPENDIX C. SEMI-CONJUGACY OF VECTOR FIELDS AND LOCAL FLOWS

Let $\mathbf{F}$ be as in (B.1). Let Y be another Banach space and let V be its open subset. Let
$$\dot{\mathbf{y}} \quad = \quad \mathbf{G}(\mathbf{y})\,. \tag{C.1}$$
be a differential equation on V. Let $\mathbf{h}$: $U \to V$ be a C^1 mapping.

Definition C.1. *We say that $\mathbf{h} \in C^1(U, V)$ is a* semi-conjugacy *of the vector field $\mathbf{G}$ to the vector field $\mathbf{F}$ if for every $\mathbf{x} \in U$:*
$$D\mathbf{h}(\mathbf{x})\, \mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{h}(\mathbf{x})). \tag{C.2}$$
The vector field $\mathbf{G}$ is called semi-conjugate *to the vector field $\mathbf{G}$ if a semi-conjugacy exists.*

If $\mathbf{h}$ is a C^1-diffeomorphism then $\mathbf{h}^{-1}$ is a semi-conjugacy of (C.1) to (B.1). In this case, $\mathbf{h}$ is called a conjugacy.

Sometimes (C.2) is called the *homological equation*, for reasons which we will not go into. We note that this equation is an infinitesimal condition which can be verified by arithmetical means not involving finding solutions to the differential equations B.1 or (C.1).

We recall that a surjective C^1 map $\mathbf{h} : U \to V$ is called a *submersion* if for every $\mathbf{x} \in M$ the Frechét derivative $D\mathbf{h}(\mathbf{x})$ is surjective. If $\mathbf{h} \in C^2(U, V)$ is a submersion then the Implicit Function Theorem implies that $\mathbf{G}$ is automatically C^1 if $\mathbf{F}$ is C^1.

Semi-conjugacies play an important role in the theory because they map solutions to solutions. More precisely, let $(\tau_-, \tau_+, \varphi)$ and (ρ_-, ρ_+, ψ) be the local flows of $\mathbf{F}$ and $\mathbf{G}$, respectively. We say that $\mathbf{h} : U \to V$ is a semi-conjugacy between these semi-flows for every $(t, \mathbf{x}) \in \mathscr{D}(\varphi)$ also $(t, \mathbf{h}(\mathbf{x})) \in \mathscr{D}(\psi)$ and
$$\psi(t, \mathbf{h}(\mathbf{x})) = \mathbf{h}(\varphi(t, \mathbf{x})). \tag{C.3}$$

While the semi-conjugacy of flows and vector fields are two different notions, (for instance, a semi-conjugacy of flows does not have to be differentiable), if $h \in C^1(U,V)$, $\mathbf{F} \in C^1(U,X)$ and $\mathbf{G} \in C^1(V,Y)$ then (C.3) and (C.2) are equivalent. For instance, differentiating (C.3) over time with the help of the Chain Rule we obtain:

$$\begin{aligned}
\mathbf{G}(\mathbf{h}(\mathbf{x})) &= \left.\frac{d\psi(t,\mathbf{h}(\mathbf{x}))}{dt}\right|_{t=0} \\
&= \left.\frac{d\mathbf{h}(\varphi(t,\mathbf{x}))}{dt}\right|_{t=0} \\
&= D\mathbf{h}(\varphi(0,\mathbf{x}))\mathbf{F}(\varphi(0,\mathbf{x})) \\
&= D\mathbf{h}(\mathbf{x})\mathbf{F}(\mathbf{x}).
\end{aligned}$$

By plugging in $t = 0$ we obtain semi-conjugacy of flows implies semi-conjugacy of vector fields. *Vice versa*, for given $\mathbf{x} \in U$ let us define $\mathbf{y}(t) = \mathbf{h}(\varphi(t,\mathbf{x}))$. Again, differentiating over t we obtain:

$$\begin{aligned}
\dot{\mathbf{y}}(t) &= D\mathbf{h}(\varphi(t,\mathbf{x}))\dot{\varphi}(t,\mathbf{x}) \\
&= D\mathbf{h}(\varphi(t,\mathbf{x}))\mathbf{F}(\varphi(t,\mathbf{x})) \\
&= \mathbf{G}(\mathbf{h}(\varphi(t,\mathbf{x}))) \\
&= \mathbf{G}(\mathbf{y}(t)).
\end{aligned}$$

Thus, $\mathbf{y}$ satisfies (C.1). We also verify easily that $\mathbf{y}(t)$ satisfies the following initial condition:

$$\mathbf{y}(0) = \mathbf{h}(\varphi(0,\mathbf{x})) = \mathbf{h}(\mathbf{x}).$$

The uniqueness of solutions implies that $\mathbf{y}(t) = \psi(t,\mathbf{h}(\mathbf{x}))$.

APPENDIX D. TIME CHANGE

We adopt the following definition:

Definition D.1. *Two vector fields $\mathbf{F},\mathbf{G} \in C^1(U,X)$ are called* time-change equivalent *if there exists a strictly-positive function $f : M \to \mathbb{R}^+$, called the* time-change*, such that $f \cdot \mathbf{G} = \mathbf{F}$.*

We note that time-change equivalence is an equivalence relation. In particular, it is symmetric because $f\mathbf{G} = \mathbf{F}$ implies $\mathbf{G} = \frac{1}{f}\mathbf{F}$.

Time-change equivalence can also be defined for local flows. There is a degree of detail in this definition that requires our attention, but the idea is that orbits are mapped to orbits, and the direction of time is preserved.

Definition D.2. *Two local flows (τ_-,τ_+,φ) and (ρ_-,ρ_+,ψ) on $U \subseteq X$ are called* time-change equivalent *if there exists a homeomorphism $\mathscr{T} : \mathscr{D}(\varphi) \to \mathscr{D}(\psi)$, of the form*

$$\mathscr{T}(t,\mathbf{x}) = (T(t,\mathbf{x}),\mathbf{x}) \tag{D.1}$$

where $T : \mathscr{D}(\varphi) \to \mathbb{R}^+$ is an increasing function of the of the first argument, such that $\psi = \varphi \circ \mathscr{T}$, i.e. for all $(t,\mathbf{x}) \in \mathscr{D}(\varphi)$

$$\psi(t,\mathbf{x}) = \varphi(T(t,\mathbf{x}),\mathbf{x}). \tag{D.2}$$

We note that as a consequence, for every $\mathbf{x} \in U$ the map

$$(\tau_-(\mathbf{x}),\tau_+(\mathbf{x})) \ni t \mapsto T(t,\mathbf{x}) \in (\rho_-(\mathbf{x}),\rho_+(\mathbf{x}))$$

is a homeomorphism of open intervals. Thus, T is a strictly increasing function of its first argument.

Lemma D.1. *Let* $\mathbf{F}, \mathbf{G} \in C^1(U, X)$*,* $f\mathbf{G} = \mathbf{F}$*, where* $f : U \to \mathbb{R}$ *is strictly positive. Let* $(\tau_-, \tau_+, \varphi)$ *and* (ρ_-, ρ_+, ψ) *be the local flows of* $\mathbf{F}$ *and* $\mathbf{G}$*, respectively. Let* $h : \mathscr{D}(\varphi) \to \overline{\mathbb{R}}$ *be defined by:*

$$h(t, \mathbf{x}) = \int_0^t f(\varphi(\tau, \mathbf{x}))\, d\tau.$$

Then there exists a well-defined function $T : \mathscr{D}(\psi) \to \overline{\mathbb{R}}$ *defined implicitly by the equation*

$$h(T(t, \mathbf{x}), \mathbf{x}) = t$$

such that for all $(t, \mathbf{x}) \in \mathscr{D}(\psi)$ *also* $(T(t, \mathbf{x}), \mathbf{x}) \in \mathscr{D}(\phi)$ *and (D.2) holds. Moreover, the map* $\mathscr{T}$ *defined by (D.1) is a diffeomorphism. In particular, the local flows are time-change equivalent.*

Proof. A major part of the proof is to show that ψ defined by (D.2) is an integral curve of the vector field $\mathbf{G}$. The following calculation does the job:

$$\begin{aligned}
\frac{d}{dt}\psi(t, \mathbf{x}) &= \left.\frac{d}{d\tau}\phi(\tau, \mathbf{x})\right|_{\tau = T(t,\mathbf{x})} \frac{d}{dt}T(t, \mathbf{x}) \\
&= \mathbf{F}(\phi(T(t, \mathbf{x}), \mathbf{x}))\ \left.\frac{1}{\frac{d}{d\tau}h(\tau, \mathbf{x})}\right|_{\tau = T(t,\mathbf{x})} \\
&= \mathbf{F}(\phi(T(t, \mathbf{x}), \mathbf{x}))\ \left.\frac{1}{f(\phi(\tau, \mathbf{x}))}\right|_{\tau = T(t,\mathbf{x})} \\
&= \mathbf{F}(\psi(t, \mathbf{x}))\, \frac{1}{f(\phi(T(t, \mathbf{x}), \mathbf{x}))} \\
&= \mathbf{F}(\psi(t, \mathbf{x}))\, \frac{1}{f(\psi(t, \mathbf{x}))} \\
&= \mathbf{G}(\psi(t, \mathbf{x})).
\end{aligned}$$

We leave the remaining details to the reader as an excercise. □

Remark D.2. This seemingly simple lemma contains the method for solving autonomous differential equations in one variable by means of quadrature.

APPENDIX E. ORBIT SEMI-CONJUGACY, ORBIT EQUIVALENCE AND ORBIT FACTORS

The notion of *orbit semi-conjugacy* unifies the notions of semi-conjugacy and time-change equivalence.

Definition E.1. *Let* $\mathbf{F} \in C^1(U, X)$ *and* $\mathbf{G} \in C^1(V, Y)$ *be two* C^1 *vector fields defined on open sets of Banach spaces* X *and* Y*. Let* $\mathbf{h} \in C^1(U, V)$ *and* $f \in C^1(U, \mathbb{R})$ *be strictly positive. We will call the pair* $(\mathbf{h}, f)$ *and* orbit semi-conjugacy *of the vector fields* $\mathbf{F}$ *and* $\mathbf{G}$ *if for every* $\mathbf{x} \in U$*:*

$$D\mathbf{h}(\mathbf{x})\, \mathbf{F}(\mathbf{x}) = f(\mathbf{x})\, \mathbf{G}(\mathbf{h}(\mathbf{x})). \tag{E.1}$$

If an orbit semi-conjugacy exists, we call the vector fields $\mathbf{F}$ *and* $\mathbf{G}$ orbit semi-conjugate.

Remark E.1. If $f \equiv 1$, we obtain the notion of ordinary semi-conjugacy. If $U = V$ and $\mathbf{h} = Id_U$, we obtain the notion of time-change equivalence.

The corresponding definition for local flows looks like this:

Definition E.2. *Two local flows* $(\tau_-, \tau_+, \varphi)$ *and* (ρ_-, ρ_+, ψ) *on open subsets* $U \subseteq X$ *and* $V \subseteq Y$ *of Banach spaces* X *and* Y *are called* orbit semi-conjugate *if there exists a homeomorphism* $\mathscr{T} : \mathscr{D}(\varphi) \to \mathscr{D}(\psi)$*, of the form*

$$\mathscr{T}(t, \mathbf{x}) = (T(t, \mathbf{x}), \mathbf{h}(\mathbf{x})) \tag{E.2}$$

where $T : \mathscr{D}(\varphi) \to \mathbb{R}^+$ *is an increasing function of the of the first argument, and* $\mathbf{h} : U \to V$ *is continuous, such that* $\psi = \varphi \circ \mathscr{T}$*, i.e. for all* $(t, \mathbf{x}) \in \mathscr{D}(\varphi)$

$$\psi(t, \mathbf{h}(\mathbf{x})) = \mathbf{h}(\varphi(T(t, \mathbf{x}), \mathbf{x})). \tag{E.3}$$

Lemma E.2. *Let us assume equation (E.1). Let* $(\tau_-, \tau_+, \varphi)$ *and* (ρ_-, ρ_+, ψ) *be the local flows of the vector fields* $\mathbf{F}$ *and* $\mathbf{G}$*, respectively. Let* $h : \mathscr{D}(\phi) \to \overline{\mathbb{R}}$ *be defined by:*

$$h(t, \mathbf{x}) = \int_0^t f(\varphi(t, \mathbf{x}))\, d\tau.$$

Let $T : \mathscr{D}(\psi) \to \mathbb{R}$ *be defined by the implicit equation* $h(T(t, \mathbf{x}), \mathbf{x}) = t$*. Then (E.3) holds. In particular, the local flows are orbit semi-conjugate.*

When $\mathbf{h}$ is a homeomorphism, we call the pair $(\mathbf{g}, f)$ and *orbit equivalence*. This is an equivalence relation. When $\mathbf{h}$ is a submersion, we call $\mathbf{G}$ an *orbit factor* of $\mathbf{F}$.

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